Eichler Cohomology for Generalized Modular Forms

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Abstract By using Stokes’s theorem, we prove an Eichler cohomology theorem for generalized modular forms with some restrictions on the relevant multiplier systems.

1 Introduction

We have undertaken the study leading to this article with the goal in mind of extending the Eichler cohomology theory([2, 20, 5, 6, 8, 14, 15, 16]) to the context of generalized modular forms (GMF’s), a subject once again receiving attention since the recent publication of [11]. We say ”once again”, since - as is not generally appreciated - Hans Petersson studied GMF’s in considerable depth in the mid- 1930’s, even including in his purview GMF’s of complex weight. (See for example, [17].)

From the outset our intention has been to derive the Eichler cohomology theorem for GMF’s by two distinct methods. The first, that of [6], employs the ”supplementary series” introduced in [7]. As it happens, the proof of our theorem by this approach has already been carried out by W. Raji [18].

Consequently, we here restrict our attention to the second method, based upon Stokes’s theorem. This approach was known to J. Lehner at least as long ago as the early 1970’s, but to our knowledge this method has not appeared in print. It turns out that the advantages of pursuing this second proof go beyond novelty in at least two respects. The more striking of these is the fact that the proof by way of Stokes’s theorem leads to a version of the Eichler cohomology theorem somewhat different from those previously known. (This new theorem reduces to the usual one when the group in question has genus 0, the case, that is, when

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all GMF’s are actually modular forms in the usual sense.) See [11, p. 6]. Secondly, the earlier method of supplementary series is successful only under a natural restriction upon the weight of the GMF’s that appear in the theorem, while the newer method employing Stokes’s theorem has no such restriction. That is to say, it yields the result for all \( k \in \mathbb{Z}, k \geq 0 \). (Compare the statement of Theorem 1 in section 3, below, with the result in [18]). The specifics are explained in section 2, following.

2 Background on Eichler Cohomology

"Generalized modular forms" (GMF’s) are defined precisely as are the usual modular forms with respect to a subgroup \( \Gamma \) of finite index in \( \Gamma(1) = SL(2, \mathbb{Z}) \), with the important exception that the definition does not require the corresponding multiplier systems (MS’s) to have absolute value one. Specifically, \( F(\tau) \) is a \textit{generalized modular form with respect to} \((\Gamma, k, v)\) provided:

(a) \( F(\tau) \) is holomorphic in \( \mathbb{H} \), the upper half-plane;

(b) \( F(\tau) \) satisfies the characteristic transformation law:

\[
F(M\tau) = v(M)(c\tau + d)^k F(\tau), \text{ for all } M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \tag{1}
\]

where \( k \in \mathbb{R} \) and \( v(M) \) is a nonzero complex number independent of \( \tau \in \mathbb{H} \);

(c) \( F(\tau) \) has a left-finite expansion of the usual exponential type at each of the parabolic points of a fundamental region \( R \) of \( \Gamma \). (See [11, section 2].)

The parameter \( k \) is the \textit{weight} of \( F \) and \( v \) is called the \textit{multiplier system} of \( F \). Note that, for \( \Gamma \) of finite index in \( \Gamma(1) \), a fundamental region \( R \) contains finitely many inequivalent parabolic cusps \( q_1, ..., q_t, t \geq 1 \). To each \( q_h \) there corresponds \( Q_h \), a parabolic element of \( \Gamma \) with the property that \( \Gamma_{q_h} = \langle Q_h, -I \rangle \), where \( \Gamma_{q_h} \) is the stabilizer of \( q_h \) in \( \Gamma \).

The \( k \)-th power in (1) is determined by the convention

\[
w^k = \left| w \right|^k e^{ik\arg w}, \tag{2}\]

where \( -\pi \leq \arg w < \pi \), for \( 0 \neq w \in \mathbb{C} \). For the most part \( k \in \mathbb{Z} \) here, in which case the argument convention is not required. If the MS \( v \) is associated with the weight \( k \in \mathbb{R} \) then \( v^{-1} \) is associated with the weight \( -k \) and thus with the weight \( 2 - k \), as well. If \( k \in \mathbb{Z} \) and \( F \neq 0 \), it follows from the transformation law (1) that

\[
v(M_1M_2) = v(M_1)v(M_2), \text{ for } M_1, M_2 \in \Gamma, \tag{3}\]

and

\[
v(-I) = (-1)^{-k} = e^{\pi ik}. \tag{4}\]
Consequently, if \( v \) is associated with weight \( k \in \mathbb{Z} \), \( \bar{v} \) is likewise associated with weight \( k \). Customarily, (4) is called the nontriviality condition for MS’s in weight \( k \). Also, adopting the terminology of [13], we call the MS unitary provided \( |v| \equiv 1 \) on \( \Gamma \).

The article [11] introduced the term "parabolic" GMF (or PGMF) for the case in which the MS has the property
\[
v(P) = 1, \text{ for all parabolic } P \text{ of trace 2 in } \Gamma. \tag{5}
\]

As is easy to see, under the assumption (3), (5) implies that \( |v(P)| = 1 \) for parabolic \( P \in \Gamma \) of trace -2. In the present context we find it convenient to introduce a more general notion, calling \( F \) weakly parabolic if \( F \) is a GMF such that its associated MS \( v \) satisfies the condition
\[
|v(P)| = 1, \text{ for all parabolic } P \in \Gamma, \tag{6}
\]
and adopting the abbreviation WPGMF for functions in this class of GMF’s. We call \( v \) weakly parabolic as well. It turns out that a good many of the results on PGMF’s in [11] hold for the larger class of WPGMF’s.

Let \( \{\Gamma, k, v\} \) denote the complex vector space of WPGMF’s associated with the parameters \( (\Gamma, k, v) \). \( C^0(\Gamma, k, v) \) and \( C^+(\Gamma, k, v) \) denote the subspaces of cusp forms and entire forms, respectively. As usual, a WPGMF is called entire if the cusp expansions contain no terms with negative exponents, and cuspidal if the cusp expansions contain exclusively terms of positive exponent. Clearly, then, \( C^0(\Gamma, k, v) \subset C^+(\Gamma, k, v) \subset \{\Gamma, k, v\} \). Henceforth, we assume that \( k \in \mathbb{Z} \) and \( v \) is a MS connected with the weight \( k \) and the group \( \Gamma \). That is, \( v \) satisfies (3) and (4), above.

Suppose that \( r \in \mathbb{Z}, r \geq 0, \) and \( a, b, c, d \in \mathbb{R} \) with \( ad - bc = 1 \). A result of G. Bol [1] states that
\[
\frac{d^{r+1}}{d\tau^{r+1}} \left( (c\tau + d)^r F \left( \frac{a\tau + b}{c\tau + d} \right) \right) = (c\tau + d)^{-r-2}F^{(r+1)} \left( \frac{a\tau + b}{c\tau + d} \right), \tag{7}
\]
provided the derivatives exist. This can be proved either by induction on \( r \) or by use of Cauchy’s integral formula.

It follows directly from (7) that for \( F \in \{\Gamma, -k, v\} \), with \( k \in \mathbb{Z}, k \geq 0, F^{(k+1)} \in \{\Gamma, k+2, v\} \). The converse of this observation is more subtle: if \( f \in \{\Gamma, k+2, v\} \), with \( k \in \mathbb{Z} \) and \( k \geq 0 \), and \( F \) is any \((k+1)\)-fold indefinite integral of \( f \), then \( F \) satisfies the transformation law:
\[
v(M)^{-1}(c\tau + d)^k F(M\tau) = F(\tau) + \rho_M(\tau), \quad z \in \mathbb{H}, \tag{8}
\]
for \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \), with \( \rho_M(\tau) \) a polynomial in \( \tau \), dependent upon \( M \), of degree at most \( k \). A function \( F \) satisfying (8), and which is meromorphic in \( \mathbb{H} \) and at the parabolic cusps of a fundamental region of \( \Gamma \), is called an Eichler integral on \( \Gamma \) of weight \(-k\) and MS \( v \), with period polynomials \( \{\rho_M : M \in \Gamma\} \).

For the slash operator defined by
\[
(F |_k^M)(\tau) = v(M)^{-1}(c\tau + d)^{-k} F(M\tau), \tag{9}
\]
with $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$, it follows that
\[
F \mid_{v}^{v} M_1 M_2 = (F \mid_{v}^{v} M_1) \mid_{v}^{v} M_2, \text{ for } M_1, M_2 \in \Gamma.
\] (10)
Furthermore, the transformation law (1) can be written more compactly as $F \mid_{v}^{v} M = F$, and (8) becomes
\[
F \mid_{v}^{v} M = F + \rho_M, \text{ for } M \in \Gamma.
\]

The latter equation leads directly to
\[
\rho_{M_1 M_2} = \rho_{M_1} \mid_{v}^{v} M_2 + \rho_{M_2}, \text{ for } M_1, M_2 \in \Gamma,
\] (11)
often called the “additive cocycle condition” with respect to $\mid_{v}^{v}$.

To introduce the ”Eichler cohomology group” we suppose that $k \geq 0$ and $\{\rho_M : M \in \Gamma\}$ is a collection of polynomials of degree at most $k$, satisfying (11); we then call $\{\rho_M : M \in \Gamma\}$ a cocycle. A coboundary is a collection $\{\rho_M : M \in \Gamma\}$ of polynomials of degree at most $k$ such that
\[
\rho_M = \rho \mid_{v}^{v} M - \rho, \text{ for } M \in \Gamma
\] (12)
with $\rho$ a fixed polynomial of degree at most $k$. Then every coboundary is a cocycle, and we define the Eichler cohomology group $H_{1}^{v}(\Gamma, P_k)$ as the quotient group: cocycles/coboundaries. Here $P_k$ denotes the vector space of polynomials of degree at most $k$.

**Remark 1.** It is easy to see that if $\{\rho_M\}$ is a collection of polynomials in $P_k$ arising as in (8), then $\{\rho_M\}$ is a cocycle; that is, $\{\rho_M\}$ satisfies (11). Note that the $(k+1)$-fold antiderivative of $f \in \{\Gamma, k+2, v\}$ occurring in (8) is determined only up to an arbitrary element of $P_k$. Thus, $f$ does not determine a unique cocycle, but rather a unique cohomology class in $H_{1}^{v}(\Gamma, P_k)$.

We introduce a natural and important subspace of $H_{1}^{v}(\Gamma, P_k)$, the space of "parabolic" cohomology classes. Suppose $\{\rho_M : M \in \Gamma\}$ is, as above, a cocycle of polynomials in $P_k$. The cocycle is called parabolic provided there exist polynomials $\rho_h$ in $P_k$ with the property
\[
\rho_{Q_h} = \rho_h \mid_{v}^{v} Q_h - \rho_h, \quad 1 \leq h \leq t,
\] (13)
where $Q_1, ..., Q_t$ are the distinct parabolic generators of $\Gamma$, corresponding to the inequivalent parabolic cusps of a fundamental region for $\Gamma$ in $\mathbb{H}$. Since a coboundary is a fortiori a parabolic cocycle, we may form the quotient group: parabolic cocycles modulo coboundaries. The resulting space $\tilde{H}_{1}^{v}(\Gamma, P_k)$, a subgroup of $H_{1}^{v}(\Gamma, P_k)$, is called the parabolic Eichler cohomology group. Note that (13) is equivalent to the following condition on the cocycle $\{\rho_M\}$.

Given a parabolic element $Q \in \Gamma$, there exists $\rho_Q^*$ in $P_k$ such that $\rho_Q = \rho_Q^* \mid_{v}^{v} Q - \rho_Q^*$. (14)
The condition (14) follows directly from (13) and the fact that each parabolic \( Q \in \Gamma \) is conjugate in \( \Gamma \) to an element of \( \Gamma_{q_h} \), for some \( h, 1 \leq h \leq t \).

The Eichler cohomology theory treats the relationships among the spaces \( H^1_v(\Gamma, P_k), \tilde{H}^1_v(\Gamma, P_k), C^0(\Gamma, k, \bar{v}), C^0(\Gamma, k, v) \) and \( C^+ (\Gamma, k, v) \). In Theorem 1, which contains the modified Eichler cohomology results that emerge in the approach applying Stokes’s theorem, the cohomology groups \( H^1, \tilde{H}^1 \) appear with suitably altered MS’s.

### 3 The Eichler cohomology theorems

The traditional Eichler cohomology theorems may be stated as follows.

**Theorem A.** Let \( k \in \mathbb{Z}, k \geq 0 \) and let \( v \) be a unitary MS in weight \( k + 2 \), with respect to \( \Gamma \). Then,

(i) the spaces \( C^+(\Gamma, k + 2, v) \oplus C^0(\Gamma, k + 2, \bar{v}) \) and \( H^1_v(\Gamma, P_k) \) are canonically isomorphic;

(ii) under the same mapping, \( C^0(\Gamma, k + 2, v) \oplus C^0(\Gamma, k + 2, \bar{v}) \) is isomorphic to \( \tilde{H}^1_v(\Gamma, P_k) \).

**Remark 2.** The significance of Theorem A lies not in the dimensionality statement, but rather in the construction of the canonical isomorphism. (See [8, pp. 611-612] and the references given there, especially [6].)

Recently, W. Raji [18] has extended Theorem A to the case of WPGMF’s, under the restriction \( k > \beta = \beta(v) \), where \( \beta > 0 \) is a constant that measures the growth of \( |v(M)| \), \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), as a function of \( a, b, c, d \). This constant is closely related to a positive constant (called \( \alpha \)) that arose initially in estimating the Fourier coefficients of entire PGMF’s [12, section 1]. (In Raji’s work the condition \( k \in \mathbb{Z} \) remains in effect, of course.) The method extends that of [6], with the additional, essential application of Eichler’s word-length estimate in Fuchsian groups of finite hyperbolic volume [3]. See also [8, p. 616]. Note that subgroups of finite index in \( \Gamma(1) \) are such groups.

Here we extend Theorem A to WPGMF’s by application of Stokes’s theorem, with the explicit results stated in Theorem 1, below. It is interesting to note that, while the proof we give for Theorem 1 (sections 4-5) is valid for unitary \( v \) (indeed we begin the proof in section 4 with an argument for this special case), when \( v \) is unitary Theorem 1 reduces to Theorem A, and we obtain nothing new in this case.

**Remark 3.** When \( \Gamma \) has genus 0, all MS’s on \( \Gamma \) are unitary [11, p. 6].

The statement of Theorem 1 requires a brief preliminary discussion of entire WPGMF’s of weight 0. It is well known that, in the case of a unitary MS \( v \), an entire WPGMF of weight 0 is constant [10], and \( v \) is thus \( \equiv 1 \) on \( \Gamma \). On the other hand, nonconstant entire WPGMF’s do exist when \( \Gamma \) has positive genus.
If $E$ is an entire WPGMF of weight 0, it is necessarily a PGMF. We denote by $v_E$ the MS of $E$, and refer to $v_E$ as an *entire* MS. A recent, as-yet-unpublished, article proves the following result essential in our considerations here [13, Theorem 4.2].

**Lemma 1.** Given $v$, a weakly parabolic MS in real weight $k$, there exists a unique entire MS $v_E$ in weight 0 such that $vv_E$ (denoted $v_u$) is unitary, again in weight $k$, of course.

**Theorem 1.** Let $k \in \mathbb{Z}$, $k \geq 0$ and $v$ a MS in weight $k+2$, with respect to $\Gamma$. Let $v_E$ be the unique entire MS in weight 0 such that $v_u = v_E$ is a unitary MS on $\Gamma$ in weight $k+2$. Let $\hat{v} = \bar{v}_u(v_{E^*})^{-1}$, with $E^*$ an arbitrary entire PGMF of weight 0 on $\Gamma$. Then,

(i) the space $C^+(\Gamma, k+2, v) \oplus C^0(\Gamma, k+2, \hat{v})$ is isomorphic to $H^1_{v_u}(\Gamma, P_k)$ under a canonical mapping;

(ii) under the same mapping, $C^0(\Gamma, k+2, v) \oplus C^0(\Gamma, k+2, \hat{v})$ is isomorphic to $\tilde{H}^1_{v_u}(\Gamma, P_k)$.

**Remark 4.** This result reduces to Theorem A if $v$ is unitary.

### 4 Proof of Theorem 1 in the unitary case

In order to emphasize the main ideas of the proof by way of Stokes’s theorem, we first establish the result for the special case in which $v$ is unitary. In this case there is no need to apply the lemma. We simplify the statement further by assuming $v_{E^*} \equiv 1$ (i.e. $E^*$ is constant). Thus we are in the situation of Theorem A:

$$| v | \equiv 1 \text{ on } \Gamma \text{ and } v_E = v_{E^*} \equiv 1, \text{ so } v_u = v \text{ and } \hat{v} = \bar{v}.$$ 

Under the assumption above, let $f \in C^+(\Gamma, k+2, v)$ and let $F$ be the $(k+1)$-st antiderivative of $f$ given explicitly, for $z \in \mathbb{H}$, by

$$F(z) = \frac{1}{k!} \int_z^i f(\tau)(z-\tau)^k d\tau, \tag{15}$$

with the path lying entirely in $\mathbb{H}$. Since $k \in \mathbb{Z}$, the integral in (15) is well defined and it is easy to show, from the transformation properties of $f$ under $\Gamma$, that

$$F|_{-k}^M = F + p_M, \text{ for } M \in \Gamma, \tag{16}$$

with

$$p_M(z) = \frac{1}{k!} \int_{M-i}^i f(\tau)(z-\tau)^k d\tau \in P_k. \tag{17}$$

Then $\alpha(f)$ is defined to be the cohomology class in $H^1_{v_u}(\Gamma, P_k)$ attached to the cocycle $\{p_M : M \in \Gamma\}$. (Note that if $F$ is replaced by any $(k+1)$-st antiderivative of $f$, the cocycle $\{p_M\}$ changes, but the cohomology class $\alpha(f)$ does not.)
Now let $g \in C^0(\Gamma, k + 2, \bar{v})$ and form the function

$$
\hat{G}(z) = \left\{ \frac{1}{k!} \int_i^z g(\tau)(\bar{z} - \tau)^k d\tau \right\}^{-},
$$

(18)

where $z \in \mathbb{H}$ and $\{\}$ $^{\text{−}}$ indicates the complex conjugate of the function inside $\{}$. A calculation analogous to the one that derives (16), (17) from the definition (15) of $F(z)$ applies here to derive

$$
\hat{G} \mid^v_{-k} M = \hat{G} + \hat{p}_M, \ M \in \Gamma,
$$

(19)

with

$$
\hat{p}_M(z) = \left\{ \frac{1}{k!} \int_{M-i}^i g(\tau)(\bar{z} - \tau)^k d\tau \right\}^{-} \in P_k.
$$

(20)

(Note that, in contrast to $\hat{G}$, $\hat{p}_M$ is holomorphic in $\mathbb{H}$.) Then $\beta(g)$ is defined to be the cohomology class in $H^1_v(\Gamma, P_k)$ attached to the cocycle $\{\hat{p}_M : M \in \Gamma\}$. This, in turn, leads to the definition of the linear map $\mu$ from $C^+(\Gamma, k + 2, v) \oplus C^0(\Gamma, k + 2, \bar{v})$ into $H^1_v(\Gamma, P_k)$ by means of

$$
\mu((f, g)) = \alpha(f) + \beta(g), \ (f \in C^+(v), g \in C^0(\bar{v})),
$$

the cohomology class of the cocycle $\{p_M\} + \{\hat{p}_M\}$. We shall show that $\mu$ is one-to-one from $C^+(v) \oplus C^0(\bar{v})$ onto $H^1_v(\Gamma, P_k)$.

With the usual definitions $(z = x + iy)$,

$$
\frac{\partial H}{\partial z} = \frac{1}{2} \left( \frac{\partial H}{\partial x} - i \frac{\partial H}{\partial y} \right),
$$

$$
\frac{\partial H}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial H}{\partial x} + i \frac{\partial H}{\partial y} \right),
$$

a straightforward calculation yields

$$
\frac{\partial \hat{G}}{\partial \bar{z}} = \frac{1}{k!} g(z)(z - \bar{z})^k.
$$

(21)

To establish that $\mu$ is one-to-one, suppose that $\mu((f, g)) = 0$ in $H^1_v(\Gamma, P_k)$. This means that there exists $p$ in $P_k$ such that

$$
p_M + \hat{p}_M = p \mid^v_{-k} M - p, \ M \in \Gamma.
$$

This, in turn, implies that $F + \hat{G} - p$ is invariant under the operator $\mid^v_{-k}$; that is to say,

$$
(F + \hat{G} - p) \mid^v_{-k} M = F + \hat{G} - p, \ M \in \Gamma.
$$

(22)

(However, note that $F + \hat{G} - p$ is not a modular form in the usual sense since $\hat{G}$ is not holomorphic, or meromorphic, in $\mathbb{H}$.)
Now, since \( g \) is holomorphic in \( \mathbb{H} \), (21) implies that
\[
\frac{\partial}{\partial z}(g(z)\hat{G}(z)) = g(z)\frac{\partial \hat{G}(z)}{\partial z} = \frac{1}{k!} \cdot |g(z)|^2 (z - \bar{z})^k.
\] (23)

On the other hand, \( g(F - p) \) is holomorphic in \( \mathbb{H} \), so that
\[
\frac{\partial}{\partial z}\{g(z)(F(z) - p(z))\} = 0, \quad z \in \mathbb{H}.
\] (24)

Let \( R \) be a fundamental region for \( \Gamma \) in \( \mathbb{H} \). (\( R \) can be chosen, for example, as a Dirichlet region, the Ford region [4] or a standard fundamental region [9, chapter 1].) By (23) and (24),
\[
\frac{1}{k!} \int \int_R |g(z)|^2 (z - \bar{z})^k \, dx \, dy = \int \int_R \frac{\partial}{\partial z}(g(z)\hat{G}(z)) \, dx \, dy = \int \int_R \{g(z)(F(z) + \hat{G}(z) - p(z))\} \, dx \, dy.
\]

Then, Stokes’s theorem applied to the right-hand side implies that
\[
\frac{1}{k!} \int \int_R |g(z)|^2 (z - \bar{z})^k \, dx \, dy = -\frac{i}{2} \int_{\partial R} g(z)\{F(z) + \hat{G}(z) - p(z)\} \, dz.
\] (25)

Observe that Stokes’s theorem, as usually stated, does not apply to the region \( R \), since the latter contains a vertical half-strip, and is thus not bounded. However, by a simple limit argument, Stokes’s theorem can be extended to apply to \( R \) in this case, since \( g \) is a cusp form and \( f \) is an entire form, so that the integrand on each side of (25) vanishes exponentially at all of the parabolic cusps, including \( i\infty \).

Now, \( g |_{k+2} M = g \) and \((F + \hat{G} - p) |_{c} M = F + \hat{G} - p\), for all \( M \) in \( \Gamma \). Thus, since \( v \) is unitary, it follows that \( g(F + \hat{G} - p) |_{c} M = g(F + \hat{G} - p) \), for \( M \) in \( \Gamma \), where \( I \) represents the identically 1 MS on \( \Gamma \). That is to say, the function \( \phi = g(F + \hat{G} - p) \) transforms under \( \Gamma \) as does a MF of weight 2 and MS \( \equiv 1 \):
\[
\phi(Mz) = (cz + d)^2 \phi(z), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.
\] (26)

Then, (26) implies that the right-hand side of (25) is 0, since the sides of \( \partial R \) are paired by transformations in \( \Gamma \). (The simplest instance of this is the pairing of the two vertical sides by the minimal translation in \( \Gamma \).)

We conclude that the left-hand side of (25) is 0, from which \( g(z) \equiv 0 \) follows. (Note that in the proof of \( g = 0 \) it is essential that the MS on the right-hand of (25) be \( \equiv 1 \).)

To complete the proof that \( \mu \) is one-to-one, it suffices to show that \( f(z) \equiv 0 \). Since \( g(z) \equiv 0 \), (18) shows that \( \hat{G}(z) \equiv 0 \); thus \((F - p) |_{c} M = F - p\), for all \( M \) in \( \Gamma \). Since \( f \) is an entire MF, it follows that \( F - p \) is an entire MF on \( \Gamma \) of weight \(-k\) and MS \( v \). But \(-k \leq 0\) and \( v \) is unitary, so that \( F - p \) is constant by a well-known result [10]. Thus, \( f = F^{(k+1)} = 0 \), and it follows that \( \mu \) is one-to-one.

The proof that the map \( \mu \) is onto \( H^1_\mathbb{R}(\Gamma, P_k) \) has been given in [6, pp. 571-574]. We do not repeat it here.

For the derivation of part \((ii)\) of theorem 1 from part \((i)\), we refer the reader to [6, pp. 574-575].
5 Proof of Theorem 1 for arbitrary MS

We return to the proof of Theorem 1, this time without the restriction that $\nu$ be unitary. Suppose then, that $\nu$ is an arbitrary MS in weight $k + 2$, with $k \in \mathbb{Z}$ and $k \geq 0$.

Let $f \in C^+(\Gamma, k + 2, \nu), g \in C^0(\Gamma, k + 2, \bar{\nu})$ and define $F, \hat{G}$, in essence, as in section 4, but modified by the appearance of the entire weight 0 PGMF’s $E, E^*$, respectively:

$$F(z) = \frac{1}{k!} \int_{\frac{i}{M-1}}^{i} f(\tau) E(\tau)(z - \tau)^k d\tau,$$

(27)

$$\hat{G}(z) = \left\{ \frac{1}{k!} \int_{\frac{i}{M-1}}^{i} g(\tau) E^*(\tau) (\bar{z} - \tau)^k d\tau \right\}^-,$$

(28)

for $z \in \mathbb{H}$. In analogy with the transformation formulae given in section 4, here we have, for $z \in \mathbb{H}$,

$$F |_{\nu_k}^{\nu_k} M = F + p_M, M \in \Gamma,$$

(29)

with

$$p_m(z) = \frac{1}{k!} \int_{\frac{i}{M-1}}^{i} f(\tau) E(\tau)(z - \tau)^k d\tau \in P_k,$$

(30)

and

$$\hat{G} |_{\nu_k}^{-\nu_k} M = \hat{G} + \hat{p}_M, M \in \Gamma,$$

(31)

where

$$\hat{p}_m(z) = \left\{ \frac{1}{k!} \int_{\frac{i}{M-1}}^{i} g(\tau) E^*(\tau) (\bar{z} - \tau)^k d\tau \right\}^- \in P_k.$$

(32)

Since $\nu_{E^*} = \bar{\nu}_a$, $\nu_{E^*}^- = \nu_a$. Thus the slash operator for the period cocycle $\{\hat{p}_M\}$ is the same as that for the period cocycle $\{p_M\}$, i.e., $|_{\nu_k}^{\nu_k}$ in both cases. This enables us to define the linear map $\mu$ from $C^+(\Gamma, k + 2, \nu) \oplus C^0(\Gamma, k + 2, \bar{\nu})$ into the Eichler cohomology group $H^1(\Gamma, P_k)$: for $f \in C^+(\Gamma, k + 2, \nu)$ and $g \in C^0(\Gamma, k + 2, \bar{\nu})$, let $\mu((f, g)) = \alpha(f) + \beta(g)$, where $\alpha(f)$ is the cohomology class in $H^1_{\nu_a}(\Gamma, P_k)$ containing the cocycle $\{p_M\}$ and $\beta(g)$ is the cohomology class containing $\{\hat{p}_M\}$.

The proof that $\mu$ is one-to-one proceeds very much as for the special case treated in section 4. In the present context, (21) is replaced by

$$\frac{\partial \hat{G}}{\partial \bar{z}} = \frac{1}{k!} g(z) E^*(\bar{z})(z - \bar{z})^k,$$

(33)

and (22) by

$$(F + \hat{G} - p) |_{\nu_k} M = F + \hat{G} - p, M \in \Gamma,$$

(34)

the latter under the assumption that $p_M + \hat{p}_M = p |_{\nu_k} M - p, M \in \Gamma$, with $p$ a fixed polynomial in $P_k$.

Furthermore, since $gE^*$ is holomorphic in $\mathbb{H}$, by (33) we have, in analogy with (23),

$$\frac{\partial}{\partial \bar{z}} \{g(z) E^*(\bar{z}) \hat{G}(z)\} = \frac{1}{k!} |g(z)|^2 |E^*(\bar{z})|^2 (z - \bar{z})^k.$$

(35)
Similarly, the equation (24) is replaced here by
\[ \frac{\partial}{\partial z} \{ g(z)E^*(z)(F(z) - p(z)) \} = 0, \ z \in \mathbb{H}. \] (36)

In the present case, the same argument which leads in section 4 to (25), by the way of Stokes’s theorem, yields
\[ \frac{1}{k!} \int \int_R |g(z)|^2 |E^*(z)|^2 (z - \bar{z})^k dx dy = -\frac{i}{2} \int_{\partial R} g(z)E^*(z)\{F(z) + \tilde{G}(z) - p(z)\} dz, \] (37)
where, as before, \( R \) is a fundamental region for \( \Gamma \). Now, for \( M \in \Gamma, g_{k+2}^E M = g \) and \( E^*|_{0}^{E^*} M = E^* \), for \( M \in \Gamma \); consequently, \( gE^*|_{k+2}^{\tilde{G}E^*} M = gE^* \), i.e. \( gE^*|_{k+2}^{u} M = gE^* \), for \( M \in \Gamma \). Together with (34), this implies that
\[ \{ gE^*(F + \tilde{G} - p) \} |_{E^*}^{|_{k}} M = \{ gE^*|_{k+2}^{u} M \} \{ (F + \tilde{G} - p) |_{E^*}^{|_{k}} M \}, \]
for \( M \in \Gamma \), since \( v_u \bar{v}_u = |v_u|^2 \equiv 1 \).

As in section 4, the right-hand side of (37) is 0. Then \( |g(z)|^2 |E^*(z)|^2 \equiv 0 \), and this implies \( g = 0 \), since, as a nontrivial PGMF of weight 0, \( E^* \) has no zero in \( \mathbb{H} \). Then, as before, \( \tilde{G} = 0 \), so that (34) becomes \((F - p) |_{E^*}^{|_{k}} M = F - p, M \in \Gamma \). But, again, this implies \( f = 0 \), and the proof that \( \mu \) is one-to-one is complete.

It remains yet to prove that \( \mu \) is onto \( H^1_{v_u}(\Gamma, P_k) \). For this it suffices to observe that the calculation done in [6, pp. 571-574] remains valid in the more general context of Theorem 1. The MS’s involved in the calculation are \( v = v_u v_{E^*}^{-1}, \bar{v} = \bar{v_u}(v_{E^*})^{-1} \) and \( v_u \). The expressions for \( \dim C^+(\Gamma, k + 2, v) \), \( \dim C^0(\Gamma, k + 2, \tilde{v}) \) and \( \dim H^1_{v_u}(\Gamma, P_k) \), as given in [6, section 4], depend only upon the values of the relevant MS’s at elliptic and parabolic generators of \( \Gamma \). But since \( E^{-1} \) and \( E^* \) are entire PGMF’s of weight 0 on \( \Gamma \), the corresponding MS’s \( v_{E^*}^{-1} \) and \( (v_{E^*})^{-1} \) are 1 at all of these generators. Thus the values of \( v, \tilde{v} \) and \( v_u \) are the same as the values of \( v_u, \bar{v_u} \) and \( v_u \), respectively, at the elliptic and parabolic generators of \( \Gamma \). It follows that \( \dim C^+(v) + \dim C^0(\tilde{v}) = \dim H^1_{v_u}(\Gamma, P_k) \) so that \( \mu \) is onto \( H^1_{v_u}(\Gamma, P_k) \). This completes the proof of Theorem 1, part (i).

**Remark 5.** The dimensions of \( C^+(v) \) and \( C^0(\tilde{v}) \) have been calculated by H. Petersson [17, Theorem 9]. The calculation of \( \dim H^1_{v_u}(\Gamma, P_k) \) follows [2, pp. 274-276]. The derivation of part (ii) from part (i) is exactly the same as the derivation of Theorem A, part (ii) from Theorem A, part (i). For details see [6, pp. 574-575].

### 6 Concluding remarks

It appears possible to extend the main results of [8] to the context of WPGMF’s, by the methods we have applied here. (These are Theorems 1 and 2, stated on p. 613 of the article.) They deal with the structure of
$H^1_v(\Gamma, \mathbb{P})$, where $\mathbb{P}$ is the space of functions $\phi$, holomorphic in $\mathbb{H}$, and subject to the growth condition: there exist $\alpha, \beta, K > 0$ such that $|\phi(z)| \leq K(|z|^\alpha + y^{-\beta})$, for $z \in \mathbb{H}$, $y = \text{Im}z$, and the relevant slash operator is $|v - k|$, with the parameter $k$ an arbitrary real number.

It would be of genuine interest as well to prove an analogue of Theorem 2 of [6], within our more general setting. This result, stated on p.567 of [6] and proved on p.574, is a restatement of Theorem A which interprets the cohomology group $H^1_v(\Gamma, P_k)$ strictly in terms of Eichler period polynomials (in $P_k$) of automorphic forms in $\{\Gamma, k + 2, v\}$. We hope to deal with these matters in a future publication.
References


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