SERIES EXPANSION OF THE PERIOD FUNCTION AND REPRESENTATIONS OF HECKE OPERATORS

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Abstract. The period polynomial of a cusp form of an integral weight plays an important role in the number theory. In this paper, we study the period function of a cusp form of real weight. We consider Hecke operators acting on cusp forms and construct their representations on the associated period functions. From this we obtain a Hecke-equivariant isomorphism between the space of cusp forms and the space of period functions. We also compute exact formulas for certain $L$-values of a Hecke eigenform using Hecke operators of period functions.

1. Introduction

For an even integer $k \geq 2$ denote the space of cusp forms of weight $k$ and trivial multiplier system on $\text{SL}(2, \mathbb{Z})$ by $S_{k,1}(1)$. For each cusp form $f \in S_{k,1}(1)$ the period polynomial of $f$ is defined by

$$
\int_{0}^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau.
$$

Then the Eichler-Shimura cohomology theory tells us that there is an isomorphism between the direct sum of two spaces of cusp forms in $S_{k,1}(1)$ and a subspace of the vector space

$$
W_k = \left\{ P \in P_{k-2}; \ P + P\big|_{2-k} S = P + P\big|_{2-k} U + P\big|_{2-k} U^2 = 0 \right\}
$$

with codimension 1. Here, $P_{k-2}$ is the space of polynomials of degree at most $k - 2$, $S$, $T$ and $U$ denote the matrices

$$
S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}
$$

in $\text{SL}(2, \mathbb{Z})$ and $\big|_{2-k}$ is the slash operator as in [2.2]. We refer to [Lan] for more details on this isomorphism. The coefficients of the period polynomial are also called periods of $f$; these periods give two additional rational structures on $S_{k,1}(1)$ besides the usual rational structure given by the rationality of Fourier coefficients. Periods are related with the critical values of $L$-functions and have been much studied by many researchers (for example, see [CZ, Kno2, KZ, Man, Za]).

Another related objects are the Hecke operators acting on cusp forms. On $S_{k,1}(1)$ the $m^{th}$ Hecke operator is represented with triangle matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ of non-negative integer entries satisfying $ad = m$ and $0 \leq b < d$ which act on $f$ by extending the slash operator accordingly.

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Their representation on period polynomials were also studied. For example, Manin constructed such representations in [Man].

In this paper, we consider cusp forms of real weight. In general the associated period function is not a polynomial any more even when the weight is half-integral. This is a big difference between the cases of integral and real weight. We discuss some aspects of such period functions in this paper. More precisely, we study two kinds of Hecke operators acting on cusp forms and period functions, respectively. With these Hecke operators we show that there is a Hecke-equivariant isomorphism between the space of cusp forms and the space of period functions. As an application, we obtain an exact formula for a certain $L$-value of a Hecke eigenform using the series expansion of its period functions.

Throughout this paper let $k \in \mathbb{R}$ be a real weight with compatible multiplier system $\chi$. We denote by $S_{k,\chi}(N)$ the space of cusp forms of weight $k$ and multiplier system $\chi$ on $\Gamma_0(N)$. If $f \in S_{k,\chi}(1)$, then $f$ has a Fourier expansion of the form

$$f(z) = \sum_{m+\kappa > 0} a(m)e^{2\pi i (m+\kappa)z},$$

where $\kappa \in [0, 1)$. Then the function

$$Pf(z) := \int_0^{i\infty} f(\tau) (\tau - z)^{k-2} d\tau$$

is the period function of the non-holomorphic Eichler integral associated with $f$.

We construct a representation of Hecke operators acting on period functions for congruence subgroups of level $N$. For integral weights Choie and Zagier [CZ] defined the action of Hecke operators on period polynomials for $\text{SL}(2, \mathbb{Z})$. Pašol and Popa [PP] generalized this approach to congruence subgroups. We construct a representation of Hecke operators acting on period functions for arbitrary weights. To do this, we consider the space $S_{vec}(N)$ of vector valued cusp forms. However, these vector valued cusp forms derived differently to the one used above. This reflects that we work here on congruence subgroups $\Gamma_0(N)$ instead of the full modular group $\text{SL}(2, \mathbb{Z})$. Let $\mu := [\text{SL}(2, \mathbb{Z}) : \Gamma_0(N)]$ and we fix representatives $\alpha_1, \ldots, \alpha_\mu$ of $\Gamma_0(N) \setminus \text{SL}(2, \mathbb{Z})$. To shorten notation we borrow the multi-index notation $\bar{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$ to refer to the right coset representatives. A change of right coset representatives are new representatives $\bar{\alpha}' = (\alpha'_1, \ldots, \alpha'_\mu)$ satisfying $\alpha'_i \in \Gamma_0(N) \alpha_i$ for all $i \in \{1, \ldots, \mu\}$.

The $m^{th}$ Hecke operator on $S_{k,\chi}(N)$ has the form (for more details see §3.3)

$$(H_{N,m}f)(z) := \left( \sum_{A \in \mathcal{X}_m} f \big|_{k,\psi} A \right)(z) = \sum_{A \in \mathcal{X}_m} \psi^{-1}(A) j(A, z)^{-k} f(Az)$$
where the sum runs through the set of upper triangle matrices

\[ X_m := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ; \ a, b, d \in \mathbb{Z}_{\geq 0}, \ ad = m, \ 0 \leq b < d \right\}. \]  

Here, we consider a map \( \psi^{-1} : X_m \to \mathbb{C} \), see §3.2 which encodes some additional scalar valued action of \( X_m \) similar to the multiplier system does for the group \( \Gamma_0(N) \). Note that we allow the case \( \psi^{-1}(A) = 0 \) for some \( A \in X_m \). We also abused/extended the slash-notation slightly to accommodate \( \psi^{-1} \). The exact definition and some explicit examples of Hecke operators \( H_{N,m} \) can be found in §3.3. A first result on Hecke operators for period functions is the following.

**Proposition 1.1.** Let \( f \in S_{k,\chi}(1) \) be a cusp form. For \( m \in \mathbb{N} \) and a suitable function \( \psi^{-1} \), the \( m \)th Hecke operator \( \tilde{H}_{1,m} \) acting on \( Pf \) is given by

\[ \tilde{H}_{1,m}(Pf) := \sum_{\lambda \in X_m} \sum_{l=1}^{L} ((Pf)|_{2-k,\chi} m_l)|_{2-k,\psi} \Lambda \]

with \( M(A0) = \sum_{l=1}^{L} m_l \in \mathbb{Z}[\text{SL}(2,\mathbb{Z})] \) a finite formal sum of matrices in \( \text{SL}(2,\mathbb{Z}) \), see Definition 3.13. The operator \( \tilde{H}_{1,m} \) satisfies

\[ \tilde{H}_{1,m}(Pf) = P(H_{1,m}f). \]

This result can be extended to \( \Gamma_0(N) \) by taking a vector valued approach. For \( f \in S_{k,\chi}(N) \) define the associated period function \( P_{\bar{\alpha}} \Pi_{\bar{\alpha}} f \) by

\[ \left[ P_{\bar{\alpha}} \Pi_{\bar{\alpha}} f \right] (z) := \int_{0}^{\infty} (f|_{0,\alpha_i}) (\tau) (\alpha_i\tau - \alpha_i z)^{k-2} d(\alpha_i \tau). \]

With weight matrices \( \omega_{\bar{\alpha}} \) depending on the weight \( k \) and multiplier system \( \chi \) and suitable extended slash notation \( \|_{\omega_{\bar{\alpha}}} \), see §2.1 we have the following result.

**Theorem 1.2.** For \( N, m \in \mathbb{N} \) and a suitable function \( \psi^{-1} \) we define the \( m \)th Hecke operator \( \tilde{H}_{N,m} \) acting on vector valued period function by

\[ \tilde{H}_{N,m}(P_{\bar{\alpha}} \Pi_{\bar{\alpha}} f)(z) := \sum_{\lambda \in X_m} \Upsilon(A)^{-1} \left( \sum_{l=1}^{L} (P_{\bar{\alpha'}} \Pi_{\bar{\alpha'}} f)\|_{\omega_{\bar{\alpha'}}} m_l \right)(Az) \]

for \( f \in S_{k,\chi}(N) \), where \( \bar{\alpha'} \) denotes the change of representatives given by (5.8). Then the operator \( \tilde{H}_{N,m} \) satisfies

\[ \tilde{H}_{N,m}(P_{\bar{\alpha}} \Pi_{\bar{\alpha}} f) = P_{\bar{\alpha}} \Pi_{\bar{\alpha}} (H_{N,m}f). \]

Furthermore, Hecke operators commute with the map \( \Pi_{\bar{\alpha}} \). (We show this in Lemma 3.18.) This immediately implies the following.
Corollary 1.3. For $f \in S_{k,\chi}(N)$

$$\tilde{H}_{N,m}(P_{\alpha}\Pi_{\alpha}f) = P_{\alpha}\Pi_{\alpha}(H_{N,m}f) = P_{\alpha}(H_{N,m}^{vec}\Pi_{\alpha}f),$$

where $H_{N,m}^{vec}$ is the $m^{th}$ Hecke operator acting on $S_{w}^{vec}(N)$ given in §3.4.

Next we prove an isomorphism between the space of cusp forms and the space of period functions. Let $P$ be the space of holomorphic functions $g$ in $\mathbb{H}$ which satisfy the growth condition

$$|g(z)| < K(|z|^r + \text{Im}(z)^{-\sigma})$$

for some positive constants $K, r$ and $\sigma$. This is the space which was introduced by Knopp in [Kno] to establish the Eichler-Shimura cohomology theory for real weights. We define

$$V_{2-k,\overline{\chi}}(P) = \left\{ h \in P; \ h + h|_{2-k,\overline{\chi}}S = 0, \ h + h|_{2-k,\overline{\chi}}U + h|_{2-k,\overline{\chi}}U^2 = 0 \right\}$$

and

$$U_{2-k,\overline{\chi}}(P) = \left\{ h - h|_{2-k,\overline{\chi}}S; \ h \in P \ 	ext{such that} \ h - h|_{2-k,\overline{\chi}}T = 0 \right\}.$$

Then we have a Hecke-equivariant isomorphism between the space of cusp forms and the space of period functions.

**Theorem 1.4.** We have a Hecke-equivariant isomorphism

$$S_{k,\chi}(1) \cong V_{2-k,\overline{\chi}}(P)/U_{2-k,\overline{\chi}}(P).$$

The map is given by the period function $f \mapsto Pf$.

As an application of the Hecke operators $\tilde{H}_{1,m}$ we obtain an exact formula for a certain $L$-value of a Hecke eigenform. For a cusp form $f \in S_{k,\chi}(1)$ with the Fourier expansion as in (1.1) its $L$-function $L(f, s)$ is given by the analytic continuation of

$$\sum_{m+\kappa>0} \frac{a(m)}{(m+\kappa)^s}.$$  

The main ingredient of the proof is the series expansion of the period function (see Lemma 5.15).

**Theorem 1.5.** Let $f \in S_{k,\chi}(1)$ be a Hecke eigenform such that $H_{1,m}f = \lambda mf$ for $m \in \mathbb{N}$. If $k > 2$, then

$$\lambda_m L(f, k-1) = i^{1-k} \frac{(2\pi)^{k-1}}{\Gamma(k-1)} \sum_{A \in X_m} \sum_{l=1}^{L} \chi(m_l)\overline{\psi(A)j(m_l, Az)^{k-2}j(A, z)^{k-2}S_f(m_lA0)},$$
where $S_f(x)$ is given by
\[
\sum_{n=0}^{\infty} \binom{k-2}{n} \left( \frac{-i}{2\pi} \right)^{k-1-n} \sum_{m+n>0} \frac{a(m)}{m+\kappa}^{k-1-n} \left( \chi(S)(-1)^n \Gamma \left( \frac{2\pi}{|x|} (m+\kappa), k-1-n \right) x^{k-2-n} + (-1)^{k-1} \Gamma(2\pi|x|(m+\kappa), k-1-n)x^n \right).
\]

Here, $\Gamma(a,n) = \int_a^{\infty} e^{-t} t^{n-1} dt$ denotes the incomplete gamma function.

This paper is organized as follows. In Sections 2 and 3, we recall basic definitions from the theory of modular forms specifically Hecke operators on cusp forms and lifts to a representation of Hecke operators on vector valued cusp forms. Section 4 discusses properties of period functions, the scalar valued cases as well as the vector valued one. Section 5 contains the proofs to our main results: Proposition 1.1 and Theorems 1.2, 1.4 and 1.5.

2. Modular forms

First we recall the definition of scalar valued modular forms. The automorphic factor $j(g,z)$ is defined by $j(g,z) := cz + d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, \mathbb{Z})$ and $z \in \mathbb{C}$. The Möbius transformation is defined by $gz = \frac{az+b}{cz+d}$ for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, \mathbb{Z})$ and $z \in \mathbb{C}$ as long as $j(g,z)$ does not vanish.

Now, consider a congruence subgroup $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$ of finite index $\mu$. Let $k \in \mathbb{R}$ denote a real weight. We call a function $\chi : \Gamma_0(N) \to \mathbb{C}$ a compatible multiplier system $\chi$ to weight $k$ if $\chi$ satisfies

1. $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma_0(N)$,
2. the compatibility condition
\[
\chi(\gamma \delta) j(\gamma, \delta, z)^k = \chi(\gamma) j(\gamma, \delta, z)^k \chi(\delta) j(\delta, z)^k
\]
for all $\gamma, \delta \in \Gamma_0(N)$ and $z \in \mathbb{H}$,
3. the non-triviality condition $\chi(-1) = e^{-\pi ik}$ with $\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We also introduce the trivial multiplier system $1$ given by
\[
1 : \Gamma_0(N) \to \mathbb{C}; \quad \gamma \mapsto 1(\gamma) := 1.
\]

We later need the extension to $1 : \mathbb{X}_m \to \mathbb{C}_{\neq 0}$, $1(A) := 1$. The slash notation now comes in two flavors, one with the multiplier system included and one without it:
\[
(f|_k, \chi)(z) := \chi(\gamma)^{-1} j(\gamma, z)^{-k} f(\gamma z) \quad \text{and} \quad (f|_k M)(z) := (\det M)^{\frac{k}{2}-1} j(M, z)^{-k} f(Mz).
\]
The first one is defined for all $\gamma \in \Gamma_0(N)$ and the last one makes sense for all $2 \times 2$ matrices $M \in \text{Mat}_+(2, \mathbb{Z})$ with positive determinant as long as $j(M, z)$ does not vanish.

The following is the precise definition of modular forms and cusp forms.

**Definition 2.1.** Suppose that a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfies

$$f|_{k, \chi} \gamma = f$$

for all $\gamma \in \Gamma_0(N)$. Moreover, if $f|_{\alpha_i}$ admits a right sided Fourier expansion of the form

$$f|_{k, \alpha_i}(z) = \sum_{n+\kappa \geq 0} a_i(n) e^{2\pi i (n+\kappa) z / \lambda_i}$$

for some $\lambda_i \in \mathbb{N}$ (the width of the cusp $\alpha_i \infty$), then $f$ is called a *modular form* of weight weight $k \in \mathbb{R}$ and compatible multiplier system $\chi$ on $\Gamma_0(N)$. A *cusp form* $f$ is a modular form that admits a right sided Fourier expansion of the form (2.4) with nonnegative $n + \kappa > 0$. The space of cusp forms is denoted by $S_{k, \chi}(N)$.

Now we introduce the notion of vector valued cusp forms.

### 2.1. Weight matrices.
We recall that $\mu \in \mathbb{N}$ denotes the index of $\Gamma_0(N)$ in $\text{SL}(2, \mathbb{Z})$ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$ denotes a vector of right representatives of the coset decomposition of $\Gamma_0(N)$ in $\text{SL}(2, \mathbb{Z})$. Following [MR1, Definition 5.1] we define the weight matrix.

**Definition 2.2.** The *weight matrix* $w : \text{SL}(2, \mathbb{Z}) \times \mathbb{H} \to \text{GL}(\mu, \mathbb{C})$ is a map satisfying the cocycle condition

$$w(\gamma \delta, z) = w(\gamma, \delta z) w(\delta, z)$$

for all $\gamma, \delta \in \text{SL}(2, \mathbb{Z})$ and $z \in \mathbb{H}$.

As example to a weight matrix with trivial multiplier system we consider the *induced representation* $\rho_0 : \text{SL}(2, \mathbb{Z}) \to \text{GL}(\mu, \mathbb{Z})$ [MR1, (5.1)], which is given by

$$\rho_0(\gamma) := \left( \delta_{\Gamma_0(N)}(\alpha_i \gamma \alpha_j^{-1}) \right)$$

for all $\gamma \in \text{SL}(2, \mathbb{Z})$ and $i, j \in \{1, \ldots, \mu\}$. The associated weight matrix $w_0(\gamma, z) = j(\gamma, z)^k \rho_0(\gamma)$ for even $k \in 2\mathbb{Z}$ is given by multiplying the scalar automorphic factor with the induced representation.

**Lemma 2.3.** The weight matrix $w_0(\gamma, z)$ defined above satisfies

$$w_0(1, z) = \text{diag}(1, \ldots, 1) \quad \text{and} \quad w_0(\gamma, z)^{-1} = w_0(\gamma^{-1}, \gamma z)$$

for all $\gamma \in \text{SL}(2, \mathbb{Z})$ and $z \in \mathbb{H}$. 
Proof. We apply (2.5) to \(1^2 = 1\) and get

\[ w_0(1, z) = w_0(1^2, z) = w_0(1, 1z) = w_0(1, z)^2. \]

This shows that \(w_0(1, z)\) has to be a projection matrix. Since by definition, \(w_0(1, z)\) is invertible, we see that \(w_0(1, z)\) has full rank. In other words, \(w_0(1, z) = \text{diag}(1, \ldots, 1)\) is the \(\mu \times \mu\) identity matrix. With the identity \(1 = \gamma^{-1} \gamma\) we get

\[ \text{diag}(1, \ldots, 1) = w_0(\gamma^{-1} \gamma, z) = w_0(\gamma^{-1}, \gamma z) w_0(\gamma, z). \]

Hence the inverse of \(w_0(\gamma, z)\) is given by \(w_0(\gamma, z)^{-1} = w_0(\gamma^{-1}, \gamma z)\).

Similar to the slash action \(\lfloor k, \chi \gamma \rfloor\) in (2.2) we define the matrix slash action. Let \(f : \mathbb{H} \to \mathbb{C}^\mu\) be a vector valued function and let \(w\) be a weight matrix. We define the matrix slash action \(\| f \|_w (\gamma)\) by

(2.8)

\[ (\| f \|_w (\gamma)) (z) := w(\gamma, z)^{-1} f(wz) \]

for all \(z \in \mathbb{H}\) and \(\gamma \in \text{SL}(2, \mathbb{Z})\). Similar to [MR1] Lemma 5.3 we show the following.

Lemma 2.4. For a multiplier system \(\chi\) of \(\Gamma_0(N)\) comparable to weight \(k\) we define the matrix valued function \(w_\alpha(\gamma, z) = (w_{i,j}(\gamma, z))_{1 \leq i,j \leq \mu}\) by

(2.9)

\[ w_{i,j}(\gamma, z) = \begin{cases} \chi(\delta_{k}(\gamma) \alpha_j^{-1}) j(\delta_{k}(\gamma) \alpha_i^{-1}, \alpha_j z)^k & \text{if } \alpha_i \gamma \alpha_j^{-1} \in \Gamma_0(N) \\ 0 & \text{if } \alpha_i \gamma \alpha_j^{-1} \notin \Gamma_0(N) \end{cases} \]

for \(\gamma \in \text{SL}(2, \mathbb{Z})\) and \(z \in \mathbb{H}\). Then \(w_\alpha\) is a weight matrix.

We may write \(w_\alpha = w\) hiding the dependency on the representatives of the right cosets.

Proof. The proof is basically the same as the one of [MR1] Lemma 5.3 just replacing the factor \(e^{-ik\text{arg}(j(\gamma, z))}\) in [MR1] with \(j(\delta_{k}(\gamma) \alpha_i^{-1}, \alpha_j z)^k\).

Remark 2.5. The weight matrix \(w\) in (2.9) resembles a permutation matrix in the following sense:

(a) In each line and column there is exactly one non-zero entry and all other entries vanish.

(b) The matrix is invertible.

The inverse matrix of \(w\) is given by

(2.10)

\[ (w(\gamma, z)^{-1})_{i,j} = \begin{cases} \chi(\delta_{k}(\gamma) \alpha_i^{-1})^{-1} j(\delta_{k}(\gamma) \alpha_i^{-1}, \alpha_j z)^{-k} & \text{if } \alpha_i \gamma^{-1} \alpha_j^{-1} \in \Gamma_0(N) \\ 0 & \text{otherwise} \end{cases} \]

for all \(z \in \mathbb{H}\) and \(\gamma \in \text{SL}(2, \mathbb{Z})\). We conclude this part defining a companion matrix to the weight matrix \(w\) in (2.9).
Definition 2.6. We define the companion weight matrix $\omega_\alpha = (\omega_{i,j})_{i,j}$ by
\[
\omega_{i,j}(g,z) := \begin{cases} 
\chi(\alpha_i \gamma \alpha_j^{-1}) j(\alpha_i \gamma \alpha_j^{-1}, \alpha_j z)^{2-k} & \text{if } \alpha_i \gamma \alpha_j^{-1} \in \Gamma_0(N) \\
0 & \text{if } \alpha_i \gamma \alpha_j^{-1} \notin \Gamma_0(N)
\end{cases}
\]
(2.11)
for all $\gamma \in \SL(2,\mathbb{Z})$ and $z \in \mathbb{H}$.

We may write $\omega_\alpha = \omega$ hiding the dependency on the representatives of the right cosets.

Example 2.7. For level $N = 1$ the $1 \times 1$-matrices $w^{-1}(A)$ and $\omega^{-1}(A)$ defined in (2.9) and (2.11) are scalar functions given by
\[
w^{-1}(\gamma, z) = \chi(\gamma)^{-1} j(\gamma, z)^{-k} \quad \text{and} \quad \omega^{-1}(\gamma, z) = \chi(\gamma) j(\gamma, z)^{k-2}
\]
for all $\gamma \in \SL(2,\mathbb{Z})$ and $z \in \mathbb{H}$.

2.2. Vector valued cusp forms. In analogy to [MRL, §5.2] we introduce vector valued cusp forms.

Definition 2.8. We call a function $\vec{f} = (f_1, \ldots, f_\mu)^{tr} : \mathbb{H} \rightarrow \mathbb{C}^\mu$ a vector valued cusp form for weight matrix $w$ if $\vec{f}$ satisfies:

(1) each component function $f_i : \mathbb{H} \rightarrow \mathbb{C}$ is holomorphic on $\mathbb{H}$,
(2) the function $\vec{f}$ satisfies the transformation law
\[
\vec{f}(gz) = w(g,z)f(z)
\]
(2.12)
for all $g \in \SL(2,\mathbb{Z})$ and $z \in \mathbb{H}$,
(3) each component function $f_i$ vanishes in the cusps.

The space of cusp forms is denoted by $S_{w}^{vec}(N)$.

We have maps between two spaces $S_{k,\chi}(N)$ and $S_{w}^{vec}(N)$.

Lemma 2.9. Maps between $S_{k,\chi}(N)$ and $S_{w}^{vec}(N)$ are given by
\[
\Pi_\alpha : S_{k,\chi}(N) \rightarrow S_{w}^{vec}(N); \quad f \mapsto (f|_0^{\alpha_1}, \ldots, f|_0^{\alpha_\mu})
\]
(2.13)
\[
\pi_\alpha : S_{w}^{vec}(N) \rightarrow S_{k,\chi}(N); \quad \vec{f} = (f_1, \ldots, f_\mu) \mapsto f_1|_0^{\alpha_1^{-1}}.
\]

We may write $\Pi_\alpha = \Pi$ and $\pi_\alpha = \pi$ hiding the dependency on the representatives.
We make sure that the enumeration is consistent, i.e., for all \( i \)
for all \( \alpha \in \Gamma \) and \( \chi \in \Gamma_0(N) \). Then

\[
\left[ (\Pi(f)\|_w g) (z) \right]_i = \left[ w(g, z)^{-1} \Pi(f)(gz) \right]_i = \sum_{j=1}^{\mu} \left( w(g, z)^{-1} \right)_{i,j} f(\alpha_j g z)
\]

\[
= \sum_{j=1}^{\mu} \delta_{\Gamma_0(N)}(\alpha_j g \alpha_i^{-1}) \chi(\alpha_j g \alpha_i^{-1})^{-1} j(\alpha_j g \alpha_i^{-1}, \alpha_i z)^{-k} f(\alpha_j g \alpha_i^{-1} \alpha_i z)
\]

where the index \( 1 \leq j_0 \leq \mu \) is uniquely determined by \( \alpha_j g \alpha_i^{-1} \in \Gamma_0(N) \). Using the modular identity (2.3) for the cusp form \( f \) we get

\[
\left[ (\Pi(f)\|_w g) (z) \right]_i = \chi(\alpha_j g \alpha_i^{-1})^{-1} j(\alpha_j g \alpha_i^{-1}, \alpha_i z)^{-k} f(\alpha_j g \alpha_i^{-1} \alpha_i z) = f(\alpha_i z) = \left[ \Pi(f) \right]_i.
\]

Hence \( \Pi(f) \) satisfies (2.12).

On the other hand, if we consider \( \bar{f} \in S^w_{\text{vec}}(N) \) and \( \gamma \in \Gamma_0(N) \), then

\[
\pi(\bar{f}\|_w \gamma)(z) = \left[ \bar{f}\|_w \gamma \right]_1(\alpha_1^{-1} z) = \left[ w(\gamma, \alpha_1^{-1} z)^{-1} \bar{f}(\gamma \alpha_1^{-1} z) \right]_1
\]

\[
= \sum_{j=1}^{\mu} \delta_{\Gamma_0(N)}(\alpha_j \gamma \alpha_1^{-1}) \chi(\alpha_j \gamma \alpha_1^{-1})^{-1} j(\alpha_j \gamma \alpha_1^{-1}, \alpha_1 \alpha_1^{-1} z)^{-k} \left[ \bar{f}(\gamma \alpha_1^{-1} z) \right]_j
\]

\[
= \chi(\alpha_1 \gamma \alpha_1^{-1})^{-1} j(\alpha_1 \gamma \alpha_1^{-1}, z)^{-k} \left[ \bar{f}(\gamma \alpha_1^{-1} z) \right]_1
\]

\[
= \chi(\alpha_1 \gamma \alpha_1^{-1})^{-1} j(\alpha_1 \gamma \alpha_1^{-1}, z)^{-k} \pi(\bar{f})(\alpha_1 \gamma \alpha_1^{-1} z)
\]

since \( \gamma \in \Gamma_0(N) \) and \( \alpha_1 \in \Gamma_0(N) \). Consider \( \pi(\bar{f}\|_w \gamma) \) again. Using (2.12) we get

\[
\pi(\bar{f}\|_w \gamma)(z) = \pi(\bar{f})(z).
\]

Hence the equation

\[
\pi(\bar{f})(z) = \chi(\alpha_1 \gamma \alpha_1^{-1})^{-1} j(\alpha_1 \gamma \alpha_1^{-1}, z)^{-k} \pi(\bar{f})(\alpha_1 \gamma \alpha_1^{-1} z)
\]

holds for all \( z \in \mathbb{H} \) and \( \gamma \in \Gamma_0(N) \). Since \( \alpha_1 \gamma \alpha_1^{-1} \) runs through all elements of \( \Gamma_0(N) \) if and only if \( \gamma \) does it, we just showed that \( \pi(\bar{f}) \) satisfies (2.3).

Let \( \alpha_1^*, \ldots, \alpha_\mu^* \in \text{SL}(2, \mathbb{Z}) \) denote another set of right coset representatives of \( \Gamma_0(N) \) in \( \text{SL}(2, \mathbb{Z}) \). We make sure that the enumeration is consistent, i.e., \( \alpha_i \) and \( \alpha_i^* \) denote the same right coset:

\[
\Gamma_0(N) \alpha_i = \Gamma_0(N) \alpha_i^*
\]

for all \( i \in \{1, \ldots, \mu\} \). Then there exist \( \gamma_i \in \Gamma_0(N) \) such that \( \gamma_i \alpha_i = \alpha_i^* \) for all \( i \). Next, we define an operation on \( S^w_{\text{vec}}(N) \) which corresponds to the change of representatives under the map \( \Pi \).
Lemma 2.10. Let $\gamma_1, \ldots, \gamma_\mu \in \Gamma_0(N)$ denote the matrices as introduced above. The operation

\begin{equation}
\bar{f}(z) \mapsto \text{diag}(\chi(\gamma_1) j(\gamma_1, \alpha_1 z)^k, \ldots, \chi(\gamma_\mu) j(\gamma_\mu, \alpha_\mu z)^k) \bar{f}(z)
\end{equation}

(2.14)

corresponds to a change of right representatives $\alpha_i \mapsto \gamma_i \alpha_i =: \alpha'_i$ in the definition of $\Pi$ in (2.13). The inverse map $\gamma_i \alpha_i \mapsto \alpha_i$ corresponds to applying the diagonal matrix \n
$$
\text{diag}(\chi(\gamma_i)^{-1} j(\gamma_i^{-1}, \gamma_i \alpha_i^i)^k) = \text{diag}(\chi(\gamma_i)^{-1} j(\gamma_i, \alpha_i^i)^{-k}).
$$

Furthermore, we have the relation

\begin{equation}
\left(\text{diag}(\chi(\gamma_1) j(\gamma_1, \alpha_1 z)^k, \ldots, \chi(\gamma_\mu) j(\gamma_\mu, \alpha_\mu z)^k) \Pi_{a'}(f)\right) \|_{w'} g = \text{diag}(\chi(\gamma_1) j(\gamma_1, \alpha_1 z)^k, \ldots, \chi(\gamma_\mu) j(\gamma_\mu, \alpha_\mu z)^k) \left(\Pi_{a}(f)\right) \|_{w} g
\end{equation}

(2.15)

for all $f \in S_{k,\chi}(N)$ and $g \in \text{SL}(2, \mathbb{Z})$, where $w$ (resp. $w'$) is the weight matrix given by (2.9) with respect to the representatives $\alpha_i$ (resp. $\alpha'_i = \gamma_i \alpha_i$).

Proof. Note that

$$
\left[\Pi_{a'}(f)(z)\right]_i = f(\alpha'_i z) = f(\gamma_i \alpha_i z) = \chi(\gamma_i) j(\gamma_i, \alpha_i z)^k f(\alpha_i z) = \chi(\gamma_i) j(\gamma_i, \alpha_i z)^k \left(\left.f \right|_0 \alpha_i\right)(z)
$$

(2.13)

$$
= \left[\text{diag}(\chi(\gamma_1) j(\gamma_1, \alpha_1 z)^k, \ldots, \chi(\gamma_\mu) j(\gamma_\mu, \alpha_\mu z)^k) \Pi_{a}(f)\right](z)
$$

$$
= \left[\Pi_{a}(\text{diag}(\chi(\gamma_1) j(\gamma_1, \cdot)^k, \ldots, \chi(\gamma_\mu) j(\gamma_\mu, \cdot)^k) f)(z)\right]_i
$$

for all $z \in \mathbb{H}$ and indices $i \in \{1, \ldots, \mu\}$. This proves (2.14). The inverse mapping with the operator \n
$$
\text{diag}(\chi(\gamma_i)^{-1} j(\gamma_i^{-1}, \gamma_i \alpha_i^i)^k) \text{ follows from } \chi(\gamma^{-1}) = \chi(\gamma)^{-1}:
$$

$$
\left[\text{diag}(\chi(\gamma_i)^{-1} j(\gamma_i^{-1}, \alpha_i^i z)^k, \ldots, \chi(\gamma_\mu)^{-1} j(\gamma_\mu^{-1}, z)^k) \Pi_{a'}(f)(z)\right]_i
$$

$$
= \chi(\gamma_i)^{-1} j(\gamma_i^{-1}, \alpha_i^i z)^k \left(\left.f \right|_0 \alpha_i^i\right)(z) = \chi(\gamma_i)^{-1} j(\gamma_i^{-1}, \gamma_i \alpha_i z)^k f(\gamma_i \alpha_i z)
$$

$$
= \chi(\gamma_i)^{-1} j(\gamma_i, \alpha_i z)^{-k} f(\gamma_i \alpha_i z) = f(\alpha_i z) = \left(\left.f \right|_0 \alpha_i\right)(z) = \left[\Pi_{a}(f)(z)\right]_i.
$$
On the other hand, if we use identities (2.10), (2.13) and (2.14), then

\[
\left[ \left( \text{diag} (\chi_{1_j} j(\gamma_1, \alpha_1 \cdot)^k, \ldots, \chi_{\mu_j} j(\gamma_\mu, \alpha_\mu \cdot)^k) \Pi(f) \right) \|_{\omega} g(z) \right]_i
\]

\[
= \sum_{j=1}^{\mu} \delta_{\Gamma_0(N)} (\alpha_j' g \alpha_j' \cdot)^{-1} \chi(\alpha_j' g \alpha_j' \cdot)^{-1} j(\alpha_j' g \alpha_j' \cdot, \alpha_j' z)^{-k} f(\alpha_j' g z)
\]

\[
= \chi(\gamma_i^{-1})^{-1} j(\gamma_i^{-1}, \gamma_i \alpha_i z)^{-k} \left[ w(g, z)^{-1} \Pi(f) (g z) \right]_i
\]

\[
= \left[ \text{diag} (\chi_{1_j} j(\gamma_1, \alpha_1 z \cdot)^k, \ldots, \chi_{\mu_j} j(\gamma_\mu, \alpha_\mu z \cdot)^k) \left( \Pi(f) \right)_{\omega} g(z) \right]_i.
\]

We just showed (2.15). □

Remark 2.11. Let \( \vec{\gamma} := (\gamma_1, \ldots, \gamma_\mu) \in \Gamma_0(N)^\mu \) denote the vector of matrices used in the change of right representatives \( \alpha_i \mapsto \gamma_i \alpha_i =: \alpha_i' \) as discussed in Lemma 2.10 by the operation defined in (2.14):

\[
(2.16) \quad \vec{f}(z) \mapsto \text{diag} (\chi_{1_j} j(\gamma_1, \alpha_1 z \cdot)^k, \ldots, \chi_{\mu_j} j(\gamma_\mu, \alpha_\mu z \cdot)^k) \vec{f}(z) =: D_{\vec{\gamma}}(\vec{f})(z)
\]

for all \( z \in \mathbb{H} \). Lemma 2.10 shows that the diagram

\[
\begin{array}{ccc}
S_{k,\chi}(N) & \longrightarrow & S_{k,\chi}(N) \\
\downarrow \Pi_{\vec{\alpha}} & & \Pi_{\vec{\alpha}'} \downarrow \\
S_{w}^{\text{vec}}(N) & \longrightarrow & S_{w}^{\text{vec}}(N)
\end{array}
\]

commutes.

3. HECKE OPERATORS ON MODULAR FORMS

In this section we introduce Hecke operators on modular forms.

3.1. Some technical results. For \( m \in \mathbb{N} \) we consider the set \( \mathbb{X}_m \) of triangle matrices defined in (1.3). We recall a few necessary technical results.

Lemma 3.1 ([Mue, Lemma 3.5]). For each \( m \in \mathbb{N} \) and \( g \in \text{SL}(2, \mathbb{Z}) \) there exists a bijective map

\[
\sigma_g : \mathbb{X}_m \rightarrow \mathbb{X}_m; \quad A \mapsto \sigma_g(A)
\]

satisfying

\[
(3.1) \quad Ag \left( \sigma_g(A) \right)^{-1} \in \text{SL}(2, \mathbb{Z})
\]

for all \( A \in \mathbb{X}_m \).
Remark 3.2. The inverse map $\sigma_g^{-1} : \mathbb{X}_m \to \mathbb{X}_m$ of $\sigma_g$, which is defined in Lemma 3.1, is given by $\sigma_g^{-1}(A) := \sigma_g^{-1}(A)$ for all $A \in \mathbb{X}_m$ and $g \in \text{SL}(2, \mathbb{Z})$. This can be seen by looking at the inverse of $A g^{-1}(\sigma_g^{-1}(A))^{-1}$:

$$
\left( A g^{-1}(\sigma_g^{-1}(A))^{-1} \right)^{-1} = \sigma_g^{-1}(A) g A^{-1} \in \text{SL}(2, \mathbb{Z}).
$$

Together with the injectivity of $\sigma_g$, which is already shown in Lemma 3.1, we conclude that $\sigma_g^{-1}$ is indeed the inverse map $\sigma_g^{-1}$ of $\sigma_g$.

In fact, Lemma 3.1 implies a bit more. Recall that we denote the right coset representatives of $\Gamma_0(N)$ in $\text{SL}(2, \mathbb{Z})$ by $\alpha_1, \ldots, \alpha_\mu$. For given $A \in \mathbb{X}_m$ and $i \in \{1, \ldots, \mu\}$, we can write $A \alpha_i$ as

$$
A \alpha_i = \gamma_{A,i} \alpha_j B
$$

with uniquely determined $\gamma_{A,i} \in \Gamma_0(N)$, $j \in \{1, \ldots, \mu\}$ and $B \in \mathbb{X}_m$. The reverse statement also holds: For given $B \in \mathbb{X}_m$ and $j \in \{1, \ldots, \mu\}$, we can write $\alpha_j B$ as $\tilde{\gamma}_{j,B} A \alpha_i = \alpha_j B$ with uniquely determined $\tilde{\gamma}_{j,B} \in \Gamma_0(N)$, $i \in \{1, \ldots, \mu\}$ and $A \in \mathbb{X}_m$. The above reasoning leads to the following.

**Definition 3.3.** For each $A \in \mathbb{X}_m$ and index $i \in \{1, \ldots, \mu\}$ consider $\sigma_{\alpha_i}(A)$ introduced in Lemma 3.1. Its defining relation by (3.1) allows us to define the maps

$$
\phi : \mathbb{X}_m \times \{1, \ldots, \mu\} \to \Gamma_0(N) \times \{1, \ldots, \mu\} \times \mathbb{X}_m; \quad (A, i) \mapsto (\phi_1(A, i), \phi_2(A, i), \phi_3(A, i))
$$

and

$$
\tilde{\phi} : \{1, \ldots, \mu\} \times \mathbb{X} \to \Gamma_0(N) \times \mathbb{X}_m \times \{1, \ldots, \mu\}; \quad (j, B) \mapsto (\tilde{\phi}_1(j, B), \tilde{\phi}_2(j, B), \tilde{\phi}_3(j, B))
$$

such that the equations

$$
A \alpha_i = \phi_1(A, i) \alpha_{\phi_2(A, i)} \phi_3(A, i) \quad \text{and} \quad \alpha_j B = \tilde{\phi}_1(j, B) \tilde{\phi}_2(j, B) \alpha_{\tilde{\phi}_3(j, B)}
$$

holds for all $A, B \in \mathbb{X}_m$ and $i, j \in \{1, \ldots, \mu\}$.

The uniqueness of the decomposition in (3.2) also implies the following.

**Lemma 3.4.** The reduced maps

$$
\mathbb{X}_m \times \{1, \ldots, \mu\} \to \{1, \ldots, \mu\} \times \mathbb{X}_m; \quad (A, i) \mapsto (\phi_2(A, i), \phi_3(A, i))
$$

and

$$
\{1, \ldots, \mu\} \times \mathbb{X} \to \mathbb{X}_m \times \{1, \ldots, \mu\}; \quad (j, B) \mapsto (\tilde{\phi}_2(j, B), \tilde{\phi}_3(j, B))
$$

are inverse to each other. Furthermore, the first component of the maps $\phi$ and $\tilde{\phi}$ satisfy

$$
\phi_1(A, i) = \left( \tilde{\phi}_1(\phi_2(A, i), \phi_3(A, i)) \right)^{-1}
$$
and
\[
\tilde{\phi}_1(j, B) = \left( \phi_1(\tilde{\phi}_2(j, B), \tilde{\phi}_3(j, B)) \right)^{-1}
\]
for all \(A, B \in \mathbb{X}_m\) and \(i, j \in \{1, \ldots, \mu\}\).

**Proof.** The lemma follows from the uniqueness of the decomposition in (3.2): Given \((A, i)\) we get unique \((B, j)\) and vice versa such that \(A \alpha_i = \gamma B \alpha_j\) for suitable unique \(\gamma \in \Gamma_0(N)\). \(\Box\)

### 3.2. Triangle matrix representations and matrix actions.

Next we define an action from element \(A \in \mathbb{X}_m\) on functions \(f : \mathbb{H} \to \mathbb{C} \neq 0\). We consider a map \(\psi^{-1} : \mathbb{X}_m \to \mathbb{C}\) such that \(A \mapsto \psi^{-1}\) induces a scalar action of \(\mathbb{X}_m\). The trivial map \(1^{-1}\) is obviously defined by \(1^{-1} : \mathbb{X}_m \to \mathbb{C}\) with \(1^{-1}(A) = 1\) for all \(A \in \mathbb{X}_m\).

**Remark 3.5.** (a) The map \(\psi^{-1}\) may also depend on the level \(N\) of the group \(\Gamma_0(N)\) and on the weight \(k\). This implicit dependency is hidden in the notation. In slight abuse of notation, its action on a scalar valued cusp form \(f \in S_{k, \chi}(N)\) is given by the slash action

\[
\left( f \left| k, \psi \right. \right)(z) = \det(A)^{k-1} \psi^{-1}(A) j(A, z)^{-k} f(Az).
\]

The above formula also indicates why we define \(\psi^{-1}(A)\). This allows us to use the slash action \(f \left| k, \psi \right.\), where this factor appears.

(b) We allow for the term \(\psi^{-1}(A)\) to vanish for some \(A\). Later, we will use this to write Hecke operators. An example of such a Hecke operator is given by Atkin and Lehner in [AL] (see also Example 3.16).

Now we consider how to define the action of a triangle matrix \(A \in \mathbb{X}_m\) on a cusp form \(f \in S_{k, \chi}(N)\). Note that

\[
\left[ \Pi(f \left| k, \psi \right. A) \right]_i
= \left( f \left| k, \psi \right. A \right) \mid_0 \alpha_i(z) = \det(A)^{k-1} \psi^{-1}(A) j(A, \alpha_i z)^{-k} f(A \alpha_i z)
= \det(A)^{k-1} \psi^{-1}(A) j(A, \alpha_i z)^{-k} f(\phi_1(A, i) \alpha \phi_2(A, i) \phi_3(A, i) z)
= \det(A)^{k-1} \psi^{-1}(A) j(A, \alpha_i z)^{-k} \chi(\phi_1(A, i)) j(\phi_1(A, i), \alpha \phi_2(A, i) \phi_3(A, i) z)^{k} f(\phi_2(A, i) \phi_3(A, i) z)
\]

for all \(A \in \mathbb{X}_m\) and \(z \in \mathbb{H}\). Since the map

\[
\mathbb{X}_m \times \{1, \ldots, \mu\} \to \{1, \ldots, \mu\} \times \mathbb{X}_m; \quad (A, i) \mapsto (\phi_2(A, i), \phi_3(A, i))
\]
is bijective, see Lemma 3.4 we can replace the image \((\phi_2(A, i), \phi_3(A, i))\) with \((j, B)\) and get

\[
\left[\Pi(f|_{k, \psi} \tilde{\phi}_2(j, B))\right]_{\tilde{\phi}_3(j, B)} (z) = \det(\tilde{\phi}_2(j, B))^{\frac{k}{2} - 1} \psi(\tilde{\phi}_2(j, B))^{-1} j(\tilde{\phi}_2(j, B), \alpha_{\tilde{\phi}_3(j, B)} z)^{-k} \\
\quad \times \chi(\tilde{\phi}_1(j, B)^{-1}) j(\tilde{\phi}_1(j, B)^{-1}, \alpha_j B z)^k f(\alpha_j B z) \\
= \det(\tilde{\phi}_2(j, B))^{\frac{k}{2} - 1} \psi(\tilde{\phi}_2(j, B))^{-1} j(\tilde{\phi}_2(j, B), \alpha_{\tilde{\phi}_3(j, B)} z)^{-k} \\
\quad \times \left[ \begin{array}{c}
\left( \chi(\tilde{\phi}_1(j, B)^{-1}) j(\tilde{\phi}_1(j, B)^{-1}, \cdot) \right)^k f(\cdot) |_{\alpha_j} \\
\end{array} \right] (Bz)
\]

We just calculated the identity

\[
(3.5) \quad \left[\Pi(f|_{k, \psi} \tilde{\phi}_2(j, B))\right]_{\tilde{\phi}_3(j, B)} (z) \\
= \det(\tilde{\phi}_2(j, B))^{\frac{k}{2} - 1} \psi(\tilde{\phi}_2(j, B))^{-1} j(\tilde{\phi}_2(j, B), \alpha_{\tilde{\phi}_3(j, B)} z)^{-k} \\
\quad \times \left[ \begin{array}{c}
\diag\left( \chi(\tilde{\phi}_1(1, B)^{-1}) j(\tilde{\phi}_1(1, B)^{-1}, \cdot) \right)^k , \\
\ldots, \chi(\tilde{\phi}_1(\mu, B)^{-1}) j(\tilde{\phi}_1(\mu, B)^{-1}, \cdot) \right)^k \Pi(f)(\cdot) \right]_{j} (Bz) \\
= \det(\tilde{\phi}_2(j, B))^{\frac{k}{2} - 1} \psi(\tilde{\phi}_2(j, B))^{-1} j(\tilde{\phi}_2(j, B), \alpha_{\tilde{\phi}_3(j, B)} z)^{-k} \left[ D_{\tilde{\phi}_1(1, B)^{-1}} \Pi(f)(\cdot) \right]_{j} (Bz)
\]

for all \(j \in \{1, \ldots, \mu\}, B \in \mathbb{X}_m\) and \(z \in \mathbb{H}\). The form of the of the right hand side indicates the three components of the above formula. The first is some kind of twisted permutation matrix which changes the \(\tilde{\phi}_3(j, B)\)th component to the \(j\)th component, since the map \(j \mapsto \tilde{\phi}_3(j, B)\) for fixed \(B \in \mathbb{X}_m\) is bijective and therefore a permutation on the set \(\{1, \ldots, \mu\}\). The second is the corresponding diagonal entry of the operation changing the representatives of the right cosets from \(\alpha_i\) to \(\alpha'_i := \tilde{\phi}_1(i, B)^{-1} \alpha_i\); the short notation \(D_{\tilde{\phi}_1(\cdot, B)^{-1}} = D_{\tilde{\phi}_1(\cdot, B)^{-1}}\) for this diagonal operator was introduced in \((2.16)\) of Remark \((2.11)\) The last component of the formula is the action of the matrix \(B \in \mathbb{X}_m\) on the vector valued form \(\Pi(f)\).

The above calculation motivates the definition of the matrix \(\Psi^{-1}(B, z)\) which is similar to the weight matrix \(w(\gamma, z)\) but is based on \(\psi\) instead on the multiplier system \(\chi\). To do so, we need one more index map.

**Lemma 3.6.** For each \(B \in \mathbb{X}_m\) the map

\[
(3.6) \quad J_B : \{1, \ldots, \mu\} \rightarrow \{1, \ldots, \mu\}; \quad j \mapsto J_B(j) := \tilde{\phi}_3(j, B)
\]

is bijective.
Proof. Let $B \in \mathbb{X}_m$ be fixed. For $j \in \{1, \ldots, \mu\}$ and $i := \tilde{\phi}_3(j, B)$ there exists an $A \in \mathbb{X}_m$ such that (3.3) holds:

$$A \alpha_i \in \Gamma_0(N) \alpha_j B.$$ 

Lemma 3.4 shows that for given $(j, B)$ the pair $(\tilde{\phi}_2(j, B), \tilde{\phi}_3(j, B))$ satisfying

$$\tilde{\phi}_2(j, B) \alpha_{\tilde{\phi}_3(j, B)} \in \Gamma_0(N) \alpha_j B$$

is unique. This implies injectivity of $j \mapsto \tilde{\phi}_3(j, B)$. The map is also surjective since the origin set and the image set of the map have the same cardinality. \hfill \Box

**Definition 3.7.** We define the matrix valued map $\Psi^{-1} : \mathbb{X}_m \times \mathbb{H} \to \text{Mat}(\mu, \mathbb{C})$ by its entries $(\Psi^{-1}(A, z))_{i,j}$ for all $i, j \in \{1, \ldots, \mu\}$ and $A \in \mathbb{X}_m$. These entries are given by

$$(3.7) \quad \Psi^{-1}_{i,j}(A) := \begin{cases} \det(\tilde{\phi}_2(j, A))^{-\frac{1}{2}} \psi(\tilde{\phi}_2(j, A))^{-1} j(\tilde{\phi}_2(j, A), \alpha_{\tilde{\phi}_3(j, A)} z)^{-k} & \text{if } i = \tilde{\phi}_3(j, A) \\ 0 & \text{if } i \neq \tilde{\phi}_3(j, A) \end{cases}$$

for all $A \in \mathbb{X}_m$, $z \in \mathbb{H}$ and $i, j \in \{1, \ldots, \mu\}$. In analogy to $\omega$ in (2.11) we define its companion $\Upsilon^{-1} : \mathbb{X}_m \times \mathbb{H} \to \text{Mat}(\mu, \mathbb{C})$ by

$$(3.8) \quad \Upsilon^{-1}_{i,j}(A) := \begin{cases} \det(\tilde{\phi}_2(j, A))^{-\frac{1}{2}} \psi(\tilde{\phi}_2(j, A))^{-1} j(\tilde{\phi}_2(j, A), \alpha_{\tilde{\phi}_3(j, A)} z)^{k-2} & \text{if } i = \tilde{\phi}_3(j, A) \\ 0 & \text{if } i \neq \tilde{\phi}_3(j, A) \end{cases}$$

for all $A \in \mathbb{X}_m$, $z \in \mathbb{H}$ and $i, j \in \{1, \ldots, \mu\}$. In analogy to $\omega$ in (2.11) we define its companion $\Upsilon^{-1} : \mathbb{X}_m \times \mathbb{H} \to \text{Mat}(\mu, \mathbb{C})$ by

Note that the inverse notation in $\Psi^{-1}(A, z)$ and $\Upsilon^{-1}(A, z)$ is a formal notation. Also, the individual matrices $\Psi^{-1}(A, z)$ and $\Upsilon^{-1}(A, z)$ may not be invertible.

**Example 3.8.** For level $N = 1$ the $1 \times 1$-matrices $\Psi^{-1}(A)$ and $\Upsilon^{-1}(A)$ are scalars given by

$$\Psi^{-1}(A) = (\det A)^{-\frac{1}{2}} \psi^{-1}(A) j(A, z)^{-k} \quad \text{and} \quad \Upsilon^{-1}(A) = (\det A)^{-\frac{1}{2}} \psi^{-1}(A) j(A, z)^{k-2}$$

for all $A \in \mathbb{X}_m$.

**Lemma 3.9.** For $A \in \mathbb{X}_m$ the matrices $\Psi^{-1}(A)$ and $\Upsilon^{-1}(A)$ defined in (3.7) and (3.8) are indeed independent of $z \in \mathbb{C}$.

Proof. Assume that $A \in \mathbb{X}_m$ has the form $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. The automorphic factor then reads as $j(A, z) = d$ which is independent of the argument $z$. This shows that appearing terms $j(\sigma_{\alpha_i^{-1}}(A), \alpha_i z)^{-k}$ and $j(A, z)^{k}$ respectively $j(\sigma_{\alpha_i^{-1}}(A), \alpha_i z)^{-k}$, $j(A, z)^{k-2}$ and $j(A, z)^{k}$ are all constants and independent of the argument $z$. Hence $\Psi^{-1}(A)$ and $\Upsilon^{-1}(A)$ are indeed matrices only depending on $A$ and the choice of representatives $\alpha_i$. \hfill \Box
Next, we define a formal matrix slash action $\|_{\Psi}$ in analogy to $\|_{w}$ defined in (2.8).

**Definition 3.10.** Let $\Psi$ denote the matrix as defined in (3.7) and let $f \in S_{k,\chi}(N)$. For any $A \in X_m$ we define

$$(\Pi_{\tilde{\alpha}}(f)_{\Psi}A)(z) := \Psi^{-1}(A) \left[ \text{diag} \left( \chi(\tilde{\phi}_1(1,A)^{-1})j(\tilde{\phi}_1(1,A)^{-1},\alpha_1 \cdot)^k, \ldots, \chi(\tilde{\phi}_1(\mu,A)^{-1})j(\tilde{\phi}_1(\mu,A)^{-1},\alpha_\mu \cdot)^k \right) \Pi_{\tilde{\alpha}}(f)(\cdot) \right](Az).$$  

(3.9)

**Remark 3.11.** With the short notation $D_{\alpha' \rightarrow \alpha^{-1}} \rightarrow \phi_1(\cdot,B)^{-1}$ for this diagonal operator introduced in (2.16) of Remark 2.11 we can rewrite the right hand side of (3.9) as

$$(\Pi_{\tilde{\alpha}}(f)_{\Psi}A)(z) = \Psi^{-1}(A) \left[ D_{\alpha' \rightarrow \alpha^{-1}} \Pi_{\tilde{\alpha}}(f) \right](Az) = \Psi^{-1}(A) \Pi_{\tilde{\alpha'}}(f)(Az).$$

**Lemma 3.12.** Let $f \in S_{k,\chi}(N)$ and let $\Psi(M,z) : \text{SL}(2,\mathbb{Z}) \times \mathbb{H} \to \mathbb{C}$ be the matrix defined in (3.7). Then

$$(\Pi_{\tilde{\alpha}}(f)_{\Psi}A)(z) = \Psi^{-1}(A) \left[ D_{\alpha' \rightarrow \alpha^{-1}} \Pi_{\tilde{\alpha}}(f) \right](Az) = \Psi^{-1}(A) \Pi_{\tilde{\alpha'}}(f)(Az).$$$$

(3.10)

**Proof.** The first equation of the lemma follows directly from the calculation in (3.5) together with the Definitions 3.7 and 3.10. For the second equation, consider (3.10). If we sum over all elements of $X_m$ we get

$$\sum_{A \in X_m} (\Pi_{\tilde{\alpha}}(f)_{\Psi}A)(z) = \sum_{A \in X_m} (\Pi(f))_{\Psi}A.$$

(3.11)

This proves (3.11) for each component $j$. □

Now we briefly recall the properties of $M(q)$ based on the discussion in [Mue §2].
Definition 3.13 ([Muc §2]). To each \( q \in [0, 1) \cap \mathbb{Q} \) we attach \( M(q) = \sum_{l=1}^{L} m_l \in \mathbb{Z}[\text{SL}(2, \mathbb{Z})] \) of the form

\[
M(q) = \left( \begin{array}{cc}
-a_0 & a_1 \\
-b_0 & b_1
\end{array} \right)^{-1} + \ldots + \left( \begin{array}{cc}
-a_{l-1} & a_l \\
-b_{l-1} & b_l
\end{array} \right)^{-1} + \ldots + \left( \begin{array}{cc}
-a_{L-1} & a_L \\
-b_{L-1} & b_L
\end{array} \right)^{-1}.
\]

Here, \( \mathbb{Z}[\text{SL}(2, \mathbb{Z})] \) denotes the set of finite linear combinations with coefficients in \( \mathbb{Z} \). The number \( L \) in Definition 3.13 depends on \( q \).

Lemma 3.14 ([Muc Lemmata 2.9–2.11]). Let \( A = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \in \mathbb{X}_m \) and consider \( M \left( \begin{array}{cc} b & d \\ 0 & a \end{array} \right) = \sum_{l=1}^{L} m_l \).

1. \( M \left( \begin{array}{cc} b & d \\ 0 & a \end{array} \right) = \sum_{l=1}^{L} \left( \begin{array}{cc} * & * \\ c_l & d_l \end{array} \right) \) and one has \( c_l \zeta + d_l > 0 \) for all \( \zeta \geq \frac{b}{a} \).

2. The matrices \( m_l A, l = 1, \ldots, L, \) contain only nonnegative integer entries.

3. The entries of the matrix \( m_l A = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \) satisfy \( a' > c' \geq 0 \) and \( d' > b' \geq 0 \).

3.3. Hecke operators for cusp forms. We introduce Hecke operator for scalar valued cusp forms of real weight.

Definition 3.15. We call a function \( \psi^{-1} : \mathbb{X}_m \to \mathbb{C} \) suitable if \( \psi^{-1} \) satisfies:

1. \( \psi^{-1} \) induces a scalar action of \( \mathbb{X}_m \).

2. \( \psi^{-1} \) is non-trivial: \( \exists A \in \mathbb{X}_m \) such that \( \psi^{-1}(A) \neq 0 \).

3. For \( f \in S_{k,\chi}(N) \) a function \( \sum_{A \in \mathbb{X}_m} f|_{k,\psi} A \) is also a cusp form in \( S_{k,\chi}(N) \).

In this case we call the operator

\[
H_{N,m} : S_{k,\chi}(N) \to S_{k,\chi}(N); \quad f \mapsto \sum_{A \in \mathbb{X}_m} f|_{k,\psi} A
\]

the \( m^{\text{th}} \) Hecke operator on \( S_{k,\chi}(N) \).

Obviously, \( H_{N,m} \) depends on the choice of \( \psi \). We will give some examples below.

Example 3.16. (1) Consider level \( N = 1 \), an even integer weight \( k \in 2\mathbb{N} \) and \( m \in \mathbb{N} \). The \( m^{\text{th}} \) Hecke operator \( H_{1,m} \) on \( S_{k,1}(1) \) is given by (for example see [Miy])

\[
H_{1,m} f = \sum_{A \in \mathbb{X}_m} f|_{k,1} A.
\]

(2) Consider the case \( N \in \mathbb{N} \), an even integer weight \( k \in 2\mathbb{N} \) with trivial multiplier system 1 and a prime number \( m \in \mathbb{N} \). We define

\[
\psi(A) := \begin{cases} 
1 & \text{if } p \nmid N, \\
1 & \text{if } p \mid N \text{ and } A \neq \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right), \\
0 & \text{if } p \mid N \text{ and } A = \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right).
\end{cases}
\]
The induced map

\[ H_{N,p} : S_{k,1}(N) \to S_{k,1}(N); \quad f \mapsto \sum_{A \in \mathcal{X}_p} f \mid_{k,\psi} A \]

is usual \( p \)th Hecke operator (for example see [AL]).

(3) Half-integral weight Hecke operators were introduced by Shimura in [Shi]. He gave a concrete example of the Hecke operator \( T_{p^2} \) on \( \Gamma_0(N) \) with \( 4 \mid N \) and prime with \( p \nmid N \) for half-integral weights.

Let \( 4 \mid N \) be a level divisible by 4 and \( k \in \frac{1}{2} \mathbb{N} \) be a half-integral weight. Let \( p \nmid N \) denote a prime which does not divide \( N \). Shimura defines the set

\[ \mathcal{X}_{p^2} := \mathcal{X}_p^2 \setminus \{ (\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}) \} =: \mathcal{A}_a \cup \mathcal{A}_b \cup \mathcal{A}_c \]

partitioned in the disjoint sets

\[ \mathcal{A}_a := \left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix}; \quad 0 \leq b < p^2 \right\}, \]

\[ \mathcal{A}_b := \left\{ \begin{pmatrix} p & h \\ 0 & p \end{pmatrix}; \quad 0 \leq h < p \right\} \quad \text{and} \]

\[ \mathcal{A}_c := \left\{ \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \]

Depending on \( p \) he defines

\[ \psi^{-1}(A) = \begin{cases} 1 \\ \left( \varepsilon_p^{-1} \left( \frac{-h}{p} \right) \right)^k \quad \text{if } A \in \mathcal{A}_a, \\ 1 \\ 0 \quad \text{if } A \in \mathcal{A}_c, \\ 1 \quad \text{if } A \notin \mathcal{X}_{p^2}, \end{cases} \]

where \( \varepsilon_p \)-constant is

\[ \varepsilon_p := \begin{cases} 1 \quad \text{if } p \equiv 1 \text{ mod } 4, \\ i \quad \text{if } p \equiv 3 \text{ mod } 4. \end{cases} \]

The multiplier system \( \chi \) is given accordingly. Then Shimura’s Hecke operator \( H_{N,p^2} \) is defined by

\[ H_{N,p^2} : S_{k,\chi}(N) \to S_{k,\chi}(N); \quad f \mapsto H_{N,p^2} f = \sum_{A \in \mathcal{X}_{p^2}} f \mid_{k,\psi} A. \]

### 3.4. Hecke operators for vector valued cusp forms.

We want to derive a formula for the Hecke operators acting on \( S^\text{vec}_w(N) \) with weight matrix \( w \) given in (2.9). To do so we have to write the vector valued cusp form \( \Pi(f \mid_{k,\psi} \sum_{A \in \mathcal{X}_m} A) \) in terms of certain matrix sums acting on the vector valued cusp form \( \Pi(f) \).
Consider the $i$th component of the vector valued cusp form $\Pi(H_{N,m}f)$. Since the map $\Pi$ is linear we can write

$$\Pi \left( \sum_{A \in \mathbb{X}_m} f|_{k,\psi} A \right)_i = \sum_{A \in \mathbb{X}_m} \Pi(f|_{k,\psi} A)_i.$$ 

Then, using Lemma 3.12, we can write each component with the $\|\psi$-notation and have

$$\left[ \Pi \left( \sum_{A \in \mathbb{X}_m} f|_{k,\psi} A \right) \right]_i = \sum_{A \in \mathbb{X}_m} \left[ \Pi(f)\|\psi A \right]_i.$$

This calculation motivates the following definition of Hecke operators for vector valued cusp forms.

**Definition 3.17.** For $m \in \mathbb{N}$ fix a $\Psi^{-1}$ as in (3.7). We call $\Psi^{-1}$ suitable if:

1. $\Psi^{-1}$ is non-trivial: $\exists A \in \mathbb{X}_m$ such that $\Psi^{-1}(A)$ is not the $\mu \times \mu$ zero-matrix.
2. For $\vec{f} \in S_{w}^{\text{vec}}(N)$ a vector valued function $\sum_{A \in \mathbb{X}_m} \vec{f}\|\psi A$ is also a vector valued cusp form in $S_{w}^{\text{vec}}(N)$.

In this case we call the operator

$$H^{\text{vec}}_{N,m} : S_{w}^{\text{vec}}(N) \to S_{w}^{\text{vec}}(N); \quad \vec{f} \mapsto \sum_{A \in \mathbb{X}_m} \vec{f}\|\psi A$$

the $m$th Hecke operator on $S_{w}^{\text{vec}}(N)$.

The $m$th Hecke operator $H^{\text{vec}}_{N,m}$ on $S_{w}^{\text{vec}}(N)$ corresponds to the $m$th Hecke operator $H_{N,m}$ on $S_{k,\chi}(N)$ as the following lemma shows.

**Lemma 3.18.** For all $f \in S_{k,\chi}(N)$

$$\Pi(H_{N,m}f) = H^{\text{vec}}_{N,m}(\Pi(f)).$$

**Proof.** By Lemma 3.12 and the linearity of $\Pi$ defined in (2.13)

$$\Pi(H_{N,m}f) = \sum_{A \in \mathbb{X}_m} \Pi(f|_{k,\psi} A) = \sum_{A \in \mathbb{X}_m} \Pi(f)\|_{\psi A} = H^{\text{vec}}_{N,m}(\Pi(f)).$$

**Remark 3.19.** In [Mue] the third author discussed Hecke operators for Maass cusp forms with trivial multiplier system. The situation in [Mue] was a bit simpler compared to our present discussion. For example, Maass cusp forms are invariant under group action $\Gamma_0(N)$. This allows us to skip the discussion about change of representatives induced by the elements $\gamma_{A,i}$ in (3.2).
4. Period functions

In this section we review the basic notions of period functions. Recall the definition of period functions $Pf$ in (1.2) for cusp forms $f \in S_{k,\chi}(1)$. This definition can be extended to higher level cusp forms $f \in S_{k,\chi}(N)$. The integral in (1.2) exists since the exponential decay of the cusp form $f$ near the cuspidal points $i\infty$ and 0 compensates for the growth of the part $(\tau - \bar{z})^{k-2}$ in the integrand.

Remark 4.1. There are other related integrals which associate cusp forms to period functions or period polynomials.

(a) Consider even weight $k$, trivial multiplier system 1 and level 1. For $f \in S_{k,1}(1)$ the integral

$$\int_{0}^{i\infty} f(\tau) (\tau - z)^{k-2} d\tau \in \mathbb{C}[z]$$

gives the associated period polynomial of degree less than $k-2$, see e.g. [Za].

(b) The third author considered a similar integral transform with replaced 1-form or integration kernel and studied Hecke operators on period functions associated to Maass cusp forms of zero weight, trivial multiplier system and level 1 (for example see [Mue]). This form was extended to cover the case of Maass cusp forms of real weights (and level 1) in [MR2].

A recent discussion of period polynomials can be found in [PP].

For the weight matrix $w$ is given by (2.9) and $\vec{f} \in S_{w}^{\text{vec}}(N)$ the associated vector valued period function $P\vec{f}$ is defined as the integral transform $P$ from the space $S_{w}^{\text{vec}}(N)$ to the space of functions on $\mathbb{H}$ given by

$$(4.1) \quad \left[ P_{\vec{f}} \right]_{i}(z) := \int_{0}^{i\infty} \left[ \vec{f} \right]_{i}(\tau) (\alpha_{i}\tau - \alpha_{i}\bar{z})^{k-2} d(\alpha_{i}\tau).$$

Formally, we write (4.1) as

$$(4.2) \quad (P_{\vec{f}})(z) = \int_{0}^{i\infty} \vec{f}(\tau) (\tau - \bar{z})^{k-2} d\tau$$

with $\vec{f}(\tau) (\tau - \bar{z})^{k-2} d\tau$ as a shortcut for

$$(4.3) \quad \left[ \vec{f}(\tau) (\tau - \bar{z})^{k-2} d\tau \right]_{i} = \left[ \vec{f} \right]_{i}(\tau) (\alpha_{i}\tau - \alpha_{i}\bar{z})^{k-2} d(\alpha_{i}\tau)$$

for all $i$. We may write $P_{\vec{f}} = P\vec{f}$ hiding the dependency on (the choice of representatives) $\bar{\alpha} = (\alpha_{1}, \ldots, \alpha_{\mu})$.

Remark 4.2. The definition of vector valued period functions (of real weight) follows closely [Mue, Definition 4.2] (up to some complex conjugations). This allows us to use most of the machinery which was introduced and discussed there.
We identify $\pm \infty$ with the cusp $i \infty$. The action of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ extends naturally to $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. By a simple path $L$ connecting points $z_0, z_1 \in \mathbb{H}^*$ we understand a piecewise smooth curve which lies inside $\mathbb{H}$ except possibly for the initial and end point $z_0, z_1$ and is analytic in all points $\mathbb{H}^* \setminus \mathbb{H}$. Two simple paths $L_{z_0, z_1}$ and $L'_{z_0, z_1}$ are always homotopic. A path $L$ connecting points $z_0, z_1 \in \mathbb{H}^*$ is given by the union of finitely many simple paths $L_n, n = 1, \ldots, N$ connecting the points $z_{0,n}, z_{1,n} \in \mathbb{H}^*$ such that $z_{0,1} = z_0, z_{1,n} = z_{0,n+1}$ and $z_{1,N} = z_1$. For distinct $z_0, z_1 \in \mathbb{H}^* \setminus \mathbb{H}$ the standard path $L_{z_0, z_1}$ is the geodesic connecting $z_0$ and $z_1$. A standard path $L$ is also a simple path (for more details see [Lan]).

For cusp forms $f \in S_{k,\chi}(N)$ it is known that the integral transformations $Pf$ satisfy (see the proof of Lemma 2.2 in [KM])

\[
\chi(\gamma^{-1})^{-1} j(\gamma^{-1}, z)^{k-2} Pf(\gamma^{-1}z) = \int_{L_{\gamma_0, \gamma_{\infty}}} f(\tau) (\tau - \bar{z})^{k-2} \, d\tau.
\]

for all $\gamma \in \Gamma_0(N)$ and $z \in \mathbb{H}$. Similarly we find for the vector valued case

\[
\omega(g^{-1}, z)^{-1} P\vec{f}(g^{-1}z) = \int_{L_{g_0, g_{\infty}}} \vec{f}(\tau) (\tau - \bar{z})^{k-2} \, d\tau
\]

for $\vec{f} \in S_{w}(N), g \in \text{SL}(2, \mathbb{Z})$ and $z \in \mathbb{H}$. Here, $\omega$ denotes the companion weight matrix to $w$ defined in (2.11).

5. PROOF OF THE MAIN THEOREMS

To prove Proposition 1.1 and Theorem 1.2 we recall a few results from [Mue].

**Lemma 5.1** ([Mue, Lemma 5.3]). For a closed 1-form $\theta$ on $\mathbb{H}$ such that $\int_L^{} \theta$ exists for all simple paths $L$ in $\mathbb{H}^*$

\[
\int_{L_{q, \infty}}^{} \theta = \int_{L_{q, q'}}^{} \theta + \int_{L_{q', \infty}}^{} \theta
\]

for all $q, q' \in \mathbb{Q}$.

**Lemma 5.2** ([Mue, Lemma 5.1]). For rational $q \in [0, 1)$ put $M(q) = \sum_l m_l$ as in Definition 3.13. The two paths $L_{q, \infty}$ and $\bigcup_l L_{m_l^{-1}0, m_l^{-1}\infty}$ have the same initial and end point.

The diagrams in Figure 1, taken from [Mue, Figure 1], illustrate the integration paths used in Lemmas 5.1 and 5.2.

First consider the Hecke operator $H_{1, m}$ acting on $f \in S_{k,\chi}(1)$. To lift this operator to an operator on period functions $Pf$ we have to understand how $f\big|_{k,\psi} A$ can be translated to period functions for $A \in \mathbb{X}_m$. 
Figure 1. Diagram (a) illustrates the paths used in Lemma 5.1. Diagram (b) illustrates the paths $L_{q,\infty}$ and $\bigcup_l L_{m_l^{-1},m_l^{-1}\infty}$ used in Lemma 5.2.

Lemma 5.3. For $A \in X_m$ and $M(A0) = \sum_{l=1}^{L} m_l \in \mathbb{Z}[\text{SL}(2,\mathbb{Z})]$

\[
\sum_{l=1}^{L} \left( Pf|_{2-k,A} m_l \right) \big|_{2-k,\overline{\psi}} A(z) \\
= (\det A)^{-\frac{k}{2}} \overline{\psi}^{-1}(A) j(A, z)^{k-2} \int_{L_{A0, A\infty}} f(\tau) (\tau - A\bar{z})^{k-2} d\tau
\]

for all $z \in \mathbb{H}$.

Proof. By (4.4) we find

\[
\int_{L_{m_l^{-1},0,m_l^{-1}\infty}} f(\tau) (\tau - A\bar{z})^{k-2} d\tau = \chi(m_l)^{-1} j(m_l, Az)^{k-2} Pf(m_l Az)
\]

for $l \in \{1, \ldots, L\}$. With the help of the slash action in (2.2) and the extension introduced in (3.4)

\[
(Pf|_{2-k,\overline{\chi}} m_l) \big|_{2-k,\overline{\psi}} A(z) = (\det A)^{-\frac{k}{2}} \overline{\psi}^{-1}(A) j(A, z)^{k-2} \left( Pf|_{2-k,\overline{\chi}} m_l \right)(Az) \\
= (\det A)^{-\frac{k}{2}} \overline{\psi}^{-1}(A) j(A, z)^{k-2} \chi(m_l)^{-1} j(m_l, Az)^{k-2} Pf(m_l Az) \\
= (\det A)^{-\frac{k}{2}} \overline{\psi}^{-1}(A) j(A, z)^{k-2} \int_{L_{m_l^{-1},0,m_l^{-1}\infty}} f(\tau) (\tau - A\bar{z})^{k-2} d\tau
\]
for all \( l \in \{1, \ldots, L\} \). Using Lemma 5.1 and the structure of the matrices \( m_l \), see (3.12), we may sum the integrals on the right hand side and get

\[
\sum_{l=1}^{L} \left( Pf|_{2-k,\chi} m_l \right)|_{2-k,\psi} A(z) = (\det A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{k-2} \frac{\sum_{l=1}^{L} \int_{L^{-1}m^{-1}_{l}0, m^{-1}_{l}1, \infty} f(\tau) (\tau - A\bar{z})^{k-2} d\tau}{L^{-1}m^{-1}_{L}L^{-1}_{0, \infty}}
\]

\[
= (\det A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{k-2} \int_{L^{-1}m^{-1}_{L}0, \infty} f(\tau) (\tau - A\bar{z})^{k-2} d\tau
\]

\[
= (\det A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{k-2} \int_{L^{-1}A_{0}, A_{\infty}} f(\tau) (\tau - A\bar{z})^{k-2} d\tau,
\]

where we used \( L_{A_0, A_{\infty}} = L^{-1}_{m^{-1}_{L}0, \infty} \) in the last step. \( \square \)

The vector valued case follows similarly.

**Lemma 5.4.** Let \( w \) denote a weight matrix as constructed in (2.9) and \( \omega \) its companion given in (2.11). Take a vector valued cusp form \( \vec{f} \in S^{vec}(N) \) and consider the associated period function \( P\vec{f} \) as defined in (4.1). For \( A \in X_m \) and \( M(A0) = \sum_{l=1}^{L} m_l \in \mathbb{Z}[SL(2, \mathbb{Z})] \)

\[
(5.2) \quad \left( \sum_{l=1}^{L} \left( Pf|_{\omega} m_l \right) \right)(Az) = \int_{L_{A_0, A_{\infty}}} \vec{f}(\tau) (\tau - A\bar{z})^{k-2} d\tau
\]

for all \( z \in \mathbb{H} \).

**Proof.** We basically copy the proof of Lemma 5.3. By (4.5) we find

\[
\int_{L^{-1}m^{-1}_{l}0, m^{-1}_{l}1, \infty} \vec{f}(\tau) (\tau - A\bar{z})^{k-2} d\tau = \omega(m_l, Az)^{-1} P\vec{f}(m_l Az)
\]

for \( l \in \{1, \ldots, L\} \). With the help of the matrix slash notation in (2.8)

\[
(Pf|_{\omega} m_l)(Az) = \omega(m_l, Az)^{-1} P\vec{f}(m_l Az) = \int_{L^{-1}m^{-1}_{l}0, m^{-1}_{l}1, \infty} \vec{f}(\tau) (\tau - A\bar{z})^{k-2} d\tau
\]
for all \( l \in \{1, \ldots, L\} \). By Lemma 5.1 and the structure of the matrices \( m_l \), see (3.12), we may sum the integrals on the right hand side and get

\[
\sum_{l=1}^{L} (Pf|_l) m_l (Az) = \sum_{l=1}^{L} \int_{L_{m^{-1}_l,0}, m^{-1}_l, \infty} f(\tau) (\tau - Az)^k d\tau
\]

where we used \( L_{A0,A\infty} = L_{m^{-1}_L,0,\infty} \) in the last step.

The next two lemmas now deal with the integrals on the right hand side of (5.1) respectively (5.2).

**Lemma 5.5.** For \( A \in X_m \) let be \( M(\sigma_\alpha(A)0) = \sum_{l=1}^{L} m_l \in \mathbb{Z}[SL(2,\mathbb{Z})] \) as in (4.1). The following identity holds:

\[
P(\int_{k,\psi} f(A) (z) = \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{-k} \int_{L_{A0,A\infty}} f(\tau) (\tau - Az)^k d\tau.
\]

**Proof.** By the substitution \( \tau = At \), one can see that the right hand side of (5.3) is the same as

\[
\det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{-k} \int_{L_{0,\infty}} f(At) (At - Az)^k dAt
\]

If we use the same argument as in the proof of Lemma 2.2 in [KM], then one can check that (5.4) is equal to

\[
\det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z)^{-k} \int_{L_{0,\infty}} f(At) \left( \det(A) \frac{t - z}{j(A, t) j(A, \bar{z})} \right)^{-k} \det A \frac{dt}{j(A, t)^2}
\]

By exploiting the extended slash notation in (3.4) we see that (5.5) equals

\[
\int_{L_{0,\infty}} \left( \det(A)^{\frac{k}{2}-1} f|_{k,\psi} f(A)(t) (t - z)^k dt
\]

Then we get the desired result due to (1.2).

**Lemma 5.6.** Let \( \alpha_1, \ldots, \alpha_\mu \) be the representatives of the right cosets of \( \Gamma_0(N) \) in \( SL(2,\mathbb{Z}) \). For \( A \in X_m \) we denote new representatives \( \alpha'_i \) of the cosets which are defined by

\[
\alpha'_i(A,i) := \phi_1(A, i) \alpha_i(A,i)
\]
for all $i$ (for the definition of $\phi_1$ see Definition 3.3). The following identity holds for $f \in S_{k,\chi}(N)$

\[
(P\Pi(f|_{k,\psi} A))(z) = \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \int_{L_{\phi_2(A,i)0,\phi_3(A,i)\infty}} [\Pi_{\alpha'}f]_{\phi_2(A,i)}(\tau) \times \\
\times (\alpha'_{\phi_2(A,i)} \tau - \alpha'_{\phi_2(A,i)} \phi_3(A,i)z)^{-k} d(\alpha'_{\phi_2(A,i)} \tau).
\]

Proof. One can check that $\alpha'$ are indeed a new set of representatives respecting the enumeration of the cosets since the map $i \mapsto \phi_2(A,i)$ is bijective on the index sets for given (fixed) $A$.

By the same argument as in the proof of Lemma 2.2 in [KM], one can see that

\[
\left[ \left( P\Pi(f|_{k,\psi} A) \right)(z) \right]_i = \int_{L_{0,\infty}} \left( \Pi(f|_{k,\psi} A) \right) \left( \alpha_i \tau - \alpha_i \bar{z} \right)^{-k} d(\alpha_i \tau)
\]

\[
= \int_{L_{0,\infty}} (f|_{k,\psi} A) \left( \alpha_i \tau - \alpha_i \bar{z} \right)^{-k} d(\alpha_i \tau)
\]

\[
= \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \int_{L_{0,\infty}} f(A\alpha_i \tau) \left( \alpha_i \tau - \alpha_i \bar{z} \right)^{-k} d(\alpha_i \tau)
\]

\[
= \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \times \\
\times \int_{L_{0,\infty}} f(A\alpha_i \tau) \left( A\alpha_i \tau - A\alpha_i \bar{z} \right)^{-k} d(A\alpha_i \tau)
\]

Then by a direct computation

\[
\left[ \left( P\Pi(f|_{k,\psi} A) \right)(z) \right]_i = \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \int_{L_{0,\infty}} f(\phi_1(A,i) \alpha_{\phi_2(A,i)} \phi_3(A,i) \tau) \times \\
\times (\phi_1(A,i) \alpha_{\phi_2(A,i)} \phi_3(A,i) \tau - \phi_1(A,i) \alpha_{\phi_2(A,i)} \phi_3(A,i) \bar{z})^{-k} \times \\
\times d(\phi_1(A,i) \alpha_{\phi_2(A,i)} \phi_3(A,i) \tau)
\]

\[
= \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \int_{L_{0,\infty}} f(\alpha'_{\phi_2(A,i)} \phi_3(A,i) \tau) \times \\
\times (\alpha'_{\phi_2(A,i)} \phi_3(A,i) \tau - \alpha'_{\phi_2(A,i)} \phi_3(A,i) \bar{z})^{-k} d(\alpha'_{\phi_2(A,i)} \phi_3(A,i) \tau)
\]

\[
= \det(A)^{-\frac{k}{2}} \psi^{-1}(A) j(A, \alpha_iz)^{-k} \int_{L_{\phi_3(A,i)0,\phi_3(A,i)\infty}} [\Pi_{\alpha'}f]_{\phi_2(A,i)}(\tau) \times \\
\times (\alpha'_{\phi_2(A,i)} \tau - \alpha'_{\phi_2(A,i)} \phi_3(A,i) \bar{z})^{-k} d(\alpha'_{\phi_2(A,i)} \tau)
\]

for all $z \in \mathbb{H}$ and $i \in \{1, \ldots, \mu\}$. This shows the identity (5.7) of the lemma. \[\square\]
Lemma 5.7. Let $B \in X_m$ be an upper triangle matrix. Denote new representatives $\alpha_i'$ of the cosets which are defined by

$$\alpha_i' := \tilde{\phi}_1(j, B)^{-1} \alpha_j$$

for all $i$. Then the following identity holds for $f \in S_{k, \chi}(N)$

$$\left( P\Pi(f|_\psi \tilde{\phi}_2(j, B)) \right)_j(z) = \left[ \Sigma^{-1}(B) \int_{L_{B\infty}} (\Pi_{\alpha'} f)(\tau) (\tau - Bz)^{k-2} d\tau \right]_j$$

(5.9)

where the vector

$$\left[ \tilde{f}(\tau - Bz)^{k-2} d\tau \right]_j = [\tilde{f}]_j (\alpha'_j \tau - \alpha'_j Bz)^{k-2} d(\alpha'_j \tau)$$

is taken with respect to the new representatives $\alpha_i'$.

Proof. By Lemma 3.4 we have the tuple $(\phi_2(A, i), \phi_3(A, i)) = (j, B)$. This allows us to write the tuple $(A, i)$ as $(\tilde{\phi}_2(j, B), \tilde{\phi}_3(j, B))$. Using this substitution in (5.7) we find the identity

$$\left( P\Pi(f|_\psi \tilde{\phi}_2(j, B)) \right)_\tilde{\phi}_3(j,B)(z) = \det(\tilde{\phi}_2(j, B))^{-1} \psi(\tilde{\phi}_2(j, B))^{-1} j(\tilde{\phi}_2(j, B), \alpha_{\tilde{\phi}_2(j,B)} z)^{k-2}$$

$$\times \int_{L_{B0,B\infty}} [\Pi_{\alpha'} f]_j(\tau) (\alpha'_j \tau - \alpha'_j Bz)^{k-2} d(\alpha'_j \tau).$$

By the definition of $\tilde{f}(\tau) (\tau - Bz)^{k-2} d\tau$ in (4.3) and the defining relation of $\Sigma^{-1}(B)$ in (3.8) we conclude

$$\left( P\Pi(f|_\psi \tilde{\phi}_2(j, B)) \right)_j(z) = \left[ \Sigma^{-1}(B) \int_{L_{B\infty}} (\Pi_{\alpha'} f)(\tau) (\tau - Bz)^{k-2} d\tau \right]_j$$

with the index permutation map $J_B$ defined in (3.6). This proves (5.9). \qed

Lemma 5.8. For $f \in S_{k, \chi}(N)$

$$\left( P\Pi(\sum_{A \in X_m} f|_{k, \psi} A) \right)_j(z) = \sum_{A \in X_m} \Sigma(A)^{-1} \int_{L_{A0, A\infty}} (\Pi_{\alpha'} f)(\tau) (\tau - Az)^{k-2} d\tau,$$

(5.10)

where the notation

$$\left[ \tilde{f}(\tau - Az)^{k-2} d\tau \right]_j = [\tilde{f}]_j (\alpha'_j \tau - \alpha'_j Az)^{k-2} d(\alpha'_j \tau)$$

is taken with respect to the new representatives $\alpha_i'$ defined in (5.8).

Proof. Similar to the proof of (3.11) we realize that the map

$$X_m \to X_m; \quad A \mapsto \tilde{\phi}_2(j, A)$$

is taken with respect to the new representatives $\alpha_i'$. \qed
is a bijection for fixed \( j \in \{1, \ldots, \mu \} \). This allows us to reenumerate the summation over all elements of \( \mathbb{X}_m \): For fixed \( i \in \{1, \ldots, \mu \} \)

\[
\left[ \left( P \Pi \left( \sum_{A \in \mathbb{X}_m} f|_{k,\psi} A \right) \right) (z) \right]_j = \sum_{A \in \mathbb{X}_m} \left[ \left( P \Pi(f|_{k,\psi} A) \right) (z) \right]_j = \sum_{A \in \mathbb{X}_m} \left[ \left( P \Pi(f|_{k,\psi} \tilde{\phi}_2(j,A)) \right) (z) \right]_j
\]

\[
= \sum_{A \in \mathbb{X}_m} \left[ Y^{-1}(B) \int_{L_{B_0,B \infty}} (\Pi A^t f)(\tau) (\tau - B \bar{z})^{k-2} d\tau \right]_j.
\]

We can repeat above calculation for every \( j \) which proves (5.10).

If we combine Lemmas 5.3 and 5.5 then we derive an explicit formula for the action of the Hecke operators on the period functions for \( \text{SL}(2, \mathbb{Z}) \) induced from the action of these operators \( H_{1,m} \) on \( S_{k,\chi}(1) \) defined in (3.13) for this group. This is our Proposition 1.1.

**Proof of Proposition 1.1.** By Lemma 5.3 and Lemma 5.5 we get

\[
\tilde{H}_{1,m}(P f)(z) = \sum_{A \in \mathbb{X}_m} \sum_{l=1}^{L} \left( (P f)|_{2-k,\bar{\chi}} m_l \right) |_{2-k,\bar{\psi}} A
\]

\[
= \sum_{A \in \mathbb{X}_m} \left( \det A \right)^{-\frac{k}{2}} \psi^{-1}(A) j(A, z) (k-2) \int_{L_{A_0, A \infty}} f(\tau) (\tau - A \bar{z})^{k-2} d\tau
\]

\[
= \sum_{A \in \mathbb{X}_m} P(f|_{k,\psi} A)(z).
\]

The well definiteness of \( \tilde{H}_{1,m}(P f)(z) \) for all \( z \in \mathbb{H} \) follows from the first statement in Lemma 3.14 since \( z \in \mathbb{H} \) ensures \( j(m_l, A z) \neq 0 \).

**Remark 5.9.** It was shown in [Mue] that the set of matrices \( m_l A \) with \( M(A_0) = \sum_{l=1}^{L} m_l \) satisfies

\[
\left\{ m_l A; A \in \mathbb{X}_m, M(A_0) = \sum_{l=1}^{L} m_l \right\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a > c \geq 0, d > c \geq 0, ad - bc = m \right\}.
\]

Similarly we can combine Lemmas 5.4 and 5.8 and derive an explicit formula for the action of the Hecke operators on the period functions for \( \Gamma_0(N) \) induced from the action of these operators \( H_{1,m} \) on \( S_{k,\chi}(1) \) defined in (3.13) for this group. This is our Theorem 1.2.
Proof of Theorem 1.2. The $m$th Hecke operator $H_{n,m}$ acts on $\bar{u}$ as

$$P_{a\Pi A}(H_{n,m}f)(z) = \left( P\Pi \left( \sum_{A \in X_m} f|_{k,\psi} A \right) \right)(z)$$

$$= \sum_{A \in X_m} \Upsilon(A)^{-1} \int_{L, A_0, A_\infty} \Pi_{\alpha} f(\tau) \left( \tau - A\bar{z} \right)^{k-2} d\tau$$

$$= \sum_{A \in X_m} \Upsilon(A)^{-1} \left( \sum_{l=1}^{L} (P_{\alpha_l} \Pi_{\alpha_l} f) ||A_{\alpha_l}m|| (A\bar{z}) \right).$$

This shows formula (1.4). The well definiteness of $\bar{H}_{N,m}(P f)(z)$ for all $z \in \mathbb{H}$ follows from the first statement in Lemma 3.14 since $z \in \mathbb{H}$ ensures $m(m, Az) \neq 0$. □

To prove Theorem 1.4 we review the Eichler-Shimura cohomology theory for real weight, which was established by Knopp [Kno2]. Let $k \in \mathbb{R}$ with a compatible multiplier system $\chi$ on $SL(2, \mathbb{Z})$. Recall that $\mathcal{P}$ is the space of holomorphic functions with a certain growth condition (15). Let $Z_{2-k,\bar{\chi}}^1(\mathcal{P})$ be the set of parabolic cocycles in $\mathcal{P}$ and $B_{2-k,\bar{\chi}}^1(\mathcal{P})$ be the set of coboundaries in $\mathcal{P}$. Let $H_{2-k,\bar{\chi}}^1(\mathcal{P})$ be the quotient space

$$H_{2-k,\bar{\chi}}^1(\mathcal{P}) := Z_{2-k,\bar{\chi}}^1(\mathcal{P})/B_{2-k,\bar{\chi}}^1(\mathcal{P}).$$

This is called the Eichler-Shimura cohomology group. We define another cohomology group associated with $T$. Let

$$Z_{2-k,\bar{\chi}}^1(\mathcal{P}; T) := \{ \varphi \in Z_{2-k,\bar{\chi}}^1(\mathcal{P}); \varphi(T) = 0 \}$$

and

$$B_{2-k,\bar{\chi}}^1(\mathcal{P}; T) := \{ \varphi \in B_{2-k,\bar{\chi}}^1(\mathcal{P}); \varphi(T) = 0 \}.$$

Then let $H_{2-k,\bar{\chi}}^1(\mathcal{P}; T)$ be the quotient space

$$H_{2-k,\bar{\chi}}^1(\mathcal{P}; T) := Z_{2-k,\bar{\chi}}^1(\mathcal{P}; T)/B_{2-k,\bar{\chi}}^1(\mathcal{P}; T).$$

To prove Theorem 1.4 we need following lemmata; their proofs are based on arguments similar to those presented in [Lau].

Lemma 5.10. The natural map $H_{2-k,\bar{\chi}}^1(\mathcal{P}; T) \to H_{2-k,\bar{\chi}}^1(\mathcal{P})$ is an isomorphism.

Proof. Let $\varphi \in Z_{2-k,\bar{\chi}}^1(\mathcal{P}) \cap B_{2-k,\bar{\chi}}^1(\mathcal{P}; T)$. By the definition of $B_{2-k,\bar{\chi}}^1(\mathcal{P})$ one can see that $\varphi \in B_{2-k,\bar{\chi}}^1(\mathcal{P})$ and the natural map is injective. We see that the map is surjective by making a translation of cocycles. That is, if $\varphi \in Z_{2-k,\bar{\chi}}^1(\mathcal{P})$, then $\varphi(T) = f - f|_{2-k,\bar{\chi}} T$ for some $f \in \mathcal{P}$ since $\varphi$ is a parabolic cocycle. Let

$$\varphi'(g) := \varphi(g) - (f - f|_{2-k,\bar{\chi}} g)$$

for $g \in SL(2, \mathbb{Z})$. Then $\varphi' \in Z_{2-k,\bar{\chi}}^1(\mathcal{P})$ and

$$\varphi + B_{2-k,\bar{\chi}}^1(\mathcal{P}) = \varphi' + B_{2-k,\bar{\chi}}^2(\mathcal{P}).$$
This implies that \([\varphi] = [\varphi']\) in \(H^1_{2-k,\tilde{\chi}}(\mathcal{P})\), where \([\varphi]\) is the coset represented by \(\varphi \in Z^1_{2-k,\tilde{\chi}}(\mathcal{P})\). Moreover, \(\varphi'(T) = 0\) so \([\varphi']\) is in the image.

\[\square\]

**Lemma 5.11.** We have an isomorphism
\[H^1_{2-k,\tilde{\chi}}(\mathcal{P}; T) \cong V^1_{2-k,\tilde{\chi}}(\mathcal{P}) / U^1_{2-k,\tilde{\chi}}(\mathcal{P}).\]

**Proof.** The map is given by
\[\varphi \mapsto \varphi(S)\]
If \(\varphi \in B^1_{2-k,\tilde{\chi}}(\mathcal{P}; T)\), then \(\varphi(g) = f - f|_{2-k,\tilde{\chi}}g\) for some \(f \in \mathcal{P}\) and \(\varphi(T) = 0\). This implies that \(f - f|_{2-k,\tilde{\chi}} T = 0\). Therefore, \(\varphi(S) \in U^1_{2-k,\tilde{\chi}}(\mathcal{P})\). Conversely, if \(f \in U^1_{2-k,\tilde{\chi}}(\mathcal{P})\), then there is a cocycle \(\varphi\) which can be defined by \(\varphi(T) = 0\) and \(\varphi(S) = f\). This is possible because \(S\) and \(T\) are generators of \(SL(2,\mathbb{Z})\). We conclude
\[B^1_{2-k,\tilde{\chi}}(\mathcal{P}; T) \cong U^1_{2-k,\tilde{\chi}}(\mathcal{P}).\]

On the other hand, if \(\varphi \in Z^1_{2-k,\tilde{\chi}}(\mathcal{P}; T)\), then one can check that \(\varphi(S) \in V^1_{2-k,\tilde{\chi}}(\mathcal{P})\). Conversely, if \(f \in V^1_{2-k,\tilde{\chi}}(\mathcal{P})\), then it induces a cocycle \(\varphi\) such that \(\varphi(S) = f\) and \(\varphi(T) = 0\), so \(\varphi \in Z^1_{2-k,\tilde{\chi}}(\mathcal{P}; T)\). Therefore, \(Z^1_{2-k,\tilde{\chi}}(\mathcal{P}; T)\) is isomorphic to \(V^1_{2-k,\tilde{\chi}}(\mathcal{P})\). From this, we can deduce that the quotient space \(H^1_{2-k,\tilde{\chi}}(\mathcal{P}; T)\) is isomorphic to the quotient space \(V^1_{2-k,\tilde{\chi}}(\mathcal{P}) / U^1_{2-k,\tilde{\chi}}(\mathcal{P})\).

**Proof of Theorem 1.4.** Knopp [Knop2] proved that there is an isomorphism
\[S_{k,\chi}(1) \cong H^1_{2-k,\tilde{\chi}}(\mathcal{P}).\]
Then by Lemma 5.10 and 5.11
\[S_{k,\chi}(1) \cong H^1_{2-k,\tilde{\chi}}(\mathcal{P}; T) \cong V^1_{2-k,\tilde{\chi}}(\mathcal{P}) / U^1_{2-k,\tilde{\chi}}(\mathcal{P}).\]
By Proposition 1.11 one can see that this map is Hecke-equivariant.

To prove Theorem 1.5 we investigate a series expansion of the period function of a cusp form \(f \in S_{k,\chi}(1)\). First we define two functions \(F_1\) and \(F_2\) as follows. For \(z \in \mathbb{H}\) we define
\[
(5.13) \quad F_1(z) : = \int_0^{i|z|} f(\tau) (\tau - z)^{k-2} \, d\tau.
\]
and
\[
(5.14) \quad F_2(z) : = \int_{i|z|}^{i\infty} f(\tau) (\tau - z)^{k-2} \, d\tau.
\]
We easily see that
\[P f(z) = F_1(z) + F_2(z),\]
holds. Now we compute these two functions more explicitly.
Lemma 5.12. Let \( f \in S_{k,\chi}(1) \) be a cusp form. Then \( F_1 \) and \( F_2 \) can be written as

\[
F_1(z) = \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} (-1)^{k-2-n} \int_{i|z|+i\varepsilon}^{i|z|-i\varepsilon} f(\tau) \tau^n d\tau
\]

and

\[
F_2(z) = \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n z^n \int_{i|z|+i\varepsilon}^{i|z|-i\varepsilon} f(\tau) \tau^{k-2-n} d\tau.
\]

Proof. Note that if \(|\tau| < |z|\), then \(|\frac{\tau}{z}| < 1\). Then we have the following binomial expansion

\[
\left(1 - \frac{\tau}{z}\right)^{k-2} = \sum_{n=0}^{\infty} \binom{k-2}{n} \left(-\frac{\tau}{z}\right)^n,
\]

where \(\binom{k-2}{n} = \frac{(k-2)(k-1)\cdots(k-1-n)}{n!}\). If we use this expansion, then we obtain

\[
(k - 2)^{k-2} = \frac{1}{z^{k-2}} (-1)^{k-2} \left(1 - \frac{\tau}{z}\right)^{k-2} = (-1)^{k-2} z^{-k+2-2} \sum_{n=0}^{\infty} \binom{k-2}{n} \left(-\frac{\tau}{z}\right)^n.
\]

Recall the defining identity (5.13) of \( F_1 \). Inserting the expression of \((\tau - \frac{z}{\tau})^{k-2}\) in \( F_1(z) \), we see that

\[
F_1(z) = \lim_{\varepsilon \downarrow 0} \int_{i|z|+i\varepsilon}^{i|z|-i\varepsilon} \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} (-1)^{k-2-n} f(\tau) \tau^n d\tau.
\]

If we use the Lebesgue dominated convergence theorem, we can interchange the integral and summation and hence we obtain

\[
F_1(z) = \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} (-1)^{k-2-n} \int_{i|z|-i\varepsilon}^{i|z|+i\varepsilon} f(\tau) \tau^n d\tau.
\]

On the other hand, if \(|\tau| > |z|\), then \(|\frac{z}{\tau}| < 1\) and we have another binomial expansion

\[
\left(1 - \frac{z}{\tau}\right)^{k-2} = \sum_{n=0}^{\infty} \binom{k-2}{n} \left(-\frac{z}{\tau}\right)^n.
\]

This expansion gives the following expression

\[
(k - 2)^{k-2} = \tau^{k-2} \left(1 - \frac{z}{\tau}\right)^{k-2} = \tau^{k-2} \sum_{n=0}^{\infty} \binom{k-2}{n} \left(-\frac{z}{\tau}\right)^n.
\]

\[
= \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n z^n \tau^{k-2-n}.
\]
With this expression we obtain the following equality analogously to the case of $F_1$

$$F_2(z) = \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n z^n \int_{|z|+i\varepsilon}^{i\infty} f(\tau) \tau^{k-2-n} \, d\tau.$$ 

This completes the proof. □

Now we show that $F_1$ and $F_2$ can be written as a series using the incomplete gamma functions.

**Proposition 5.13.** Functions $F_1$ and $F_2$ can be written as

$$F_1(z) = \lim_{\varepsilon \downarrow 0} \left\{ \chi(S) \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n \left( \frac{i}{2\pi} \right)^{k-1-n} z^{k-2-n} \times \sum_{m+\kappa>0} a(m) \Gamma \left( \frac{2\pi(m+\kappa)}{|z|+\varepsilon}, k-1-n \right) \right\}$$

and

$$F_2(z) = \lim_{\varepsilon \downarrow 0} \left\{ \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^{k-1} \left( \frac{i}{2\pi} \right)^{k-1-n} z^n \times \sum_{m+\kappa>0} a(m) \Gamma \left( \frac{2\pi(m+\kappa)(|z|+\varepsilon), k-1-n} \right) \right\}.$$

**Proof.** To prove Proposition 5.13, we should compute the integral

$$G_{k-2-n}(\varepsilon) := \int_{i\varepsilon}^{i\infty} f(\tau) \tau^{k-2-n} \, d\tau$$

for $\varepsilon > 0$. By the change of variable, we see that

$$G_{k-2-n}(\varepsilon) = \int_{\varepsilon}^{\infty} f(iv) (iv)^{k-2-n} i\, dv.$$ 

By the Fourier expansion of $f$ and the Lebesgue dominated convergence theorem, we obtain

$$G_{k-2-n}(\varepsilon) = i^{k-1-n} \int_{\varepsilon}^{\infty} \sum_{m+\kappa>0} a(m) e^{-2\pi(m+\kappa)v} v^{k-2-n} \, dv$$

$$= i^{k-1-n} \sum_{m+\kappa>0} a(m) \int_{\varepsilon}^{\infty} e^{-2\pi(m+\kappa)v} v^{k-2-n} \, dv.$$
Lemma 5.14. Suppose that $k > 2$. Then

$$F_1(z) = \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} \frac{(-1)^n}{n!} \int_{1-\varepsilon}^{1+\varepsilon} f(\tau) \tau^n d\tau$$

Therefore, we get the desired result. □

If we use the incomplete gamma function, the last integral can be written as

$$G_{k-2-n}(\varepsilon) = \int_{\varepsilon}^{\infty} e^{-2\pi(m+\kappa)v} v^{k-2-n} dv$$

$$= \int_{2\pi(m+\kappa)\varepsilon}^{\infty} e^{-t} \left( \frac{t}{2\pi(m+\kappa)} \right)^{k-2-n} \frac{dt}{2\pi(m+\kappa)}$$

$$= \frac{1}{(2\pi(m+\kappa))^{k-1-n}} \Gamma(2\pi(m+\kappa)\varepsilon, k - 1 - n).$$

Therefore, we see that

$$G_{k-2-n}(\varepsilon) = \left( \frac{i}{2\pi} \right)^{k-1-n} \sum_{m+\kappa > 0} \frac{a(m) \Gamma(2\pi(m+\kappa)\varepsilon, k - 1 - n)}{(m+\kappa)^{k-1-n}}.$$

If we use Lemma 5.12 and the transformation property (2.3) applies to $S$, then

$$F_1(z) = \lim_{\varepsilon \to 0} \frac{\chi(S)}{(1)\varepsilon^{k-1-n}} \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} \int_{|z| - \varepsilon}^{1} f(\tau) \tau^{k-2-n} d\tau.$$

With the expression of $G_{k-2-n}(\varepsilon)$ as above we find

$$F_1(z) = \lim_{\varepsilon \to 0} \frac{\chi(S)}{(1)\varepsilon^{k-1-n}} \sum_{n=0}^{\infty} \binom{k-2}{n} z^{k-2-n} G_{k-2-n} \left( \frac{1}{|z| - \varepsilon} \right)$$

$$= \lim_{\varepsilon \to 0} \left\{ \chi(S) \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n \left( \frac{i}{2\pi} \right)^{k-1-n} z^{k-2-n} \right. \times \sum_{m+\kappa > 0} \frac{a(m) \Gamma(2\pi(m+\kappa)\varepsilon, k - 1 - n)}{(m+\kappa)^{k-1-n}} \right\}.$$

By the same way, we see that

$$F_2(z) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^n z^n G_{k-2-n}(|z| - +\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \left\{ \sum_{n=0}^{\infty} \binom{k-2}{n} (-1)^{k-1} \left( \frac{i}{2\pi} \right)^{k-1-n} z^n \times \sum_{m+\kappa > 0} \frac{a(m) \Gamma(2\pi(m+\kappa)(|z| - +\varepsilon), k - 1 - n)}{(m+\kappa)^{k-1-n}} \right\}.$$

Therefore, we get the desired result.
Then for sufficiently large $x$, we define the Gauß brackets 

$$\text{gcd}(x) = \frac{(k-2)(k-3) \cdots (k-2-(n-1))}{1 \cdot 2 \cdots n} = \left( \frac{k-2 \cdots k-2-(a-1)}{n \cdots n-(a-1)} \right) \cdot \left( \frac{k-2-a \cdots k-2-(n-1)}{1 \cdots 1} \right).$$

Note that $\left| \frac{k-2-a}{n-a} \right|, \ldots, \left| \frac{k-2-(n-1)}{n-a} \right|$ are less than 1 since $k-2 < a \leq k-1$. Therefore

$$\left( \frac{k-2}{n} \right) \leq \frac{k-2 \cdots k-2-(a-1)}{n \cdots n-(a-1)}.$$

Furthermore, we see that

$$\left| \int_0^{i\varepsilon} f(\tau) \tau^n d\tau \right| \leq \int_0^{\varepsilon} f(it)(it)^n idt \leq M_\varepsilon \int_0^{\varepsilon} t^n dt = M_\varepsilon \frac{\varepsilon^{n+1}}{n+1},$$

where $M_\varepsilon$ is a positive number such that $\left| f(it) \right| < M_\varepsilon$ for all $0 \leq t \leq \varepsilon$. Such a number exists because $f$ is a cusp form.

By the above analysis, we find for $F_1$

$$\left( \frac{k-2}{n} \right) z^{k-2-n} (-1)^{k-2-n} \int_0^{i|z|} f(\tau) \tau^n d\tau \leq \frac{k-2 \cdots k-2-(a-1)}{n \cdots n-(a-1)} \left| z \right|^{k-1} M_{|z|} \frac{1}{n+1} = \frac{(k-2) \cdots (k-2-(a-1)) \left| z \right|^{k-1} M_{|z|}}{2(n+1)n \cdots (n-(a-1))}.$$ 

Therefore, we see that

$$\sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) z^{k-2-n} (-1)^{k-2-n} \int_0^{i|z|} f(\tau) \tau^n d\tau$$

converges absolutely. So, we may remove the limit in (5.15).

For $F_2$ we use Lemma 5.12 and the property of $f$ that $f_{\chi, \chi} S = f$ so that we have

$$F_2(z) = \lim_{\varepsilon \downarrow 0} \chi(S) \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^{k-1} z^n \int_0^{i|z|} f(\tau) \tau^n d\tau.$$

The same arguments as used in the case of $F_1$ show that

$$\sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^{k-1} z^n \int_0^{i|z|} f(\tau) \tau^n d\tau$$

and

$$F_2(z) = \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^n z^n \int_0^{i|z|} f(\tau) \tau^n d\tau.$$
converges absolutely. This completes the proof. □

We can show that the period function of \( f \) has a series expansion.

**Lemma 5.15.** Let \( k \in \mathbb{R} \) with \( k > 2 \) and \( r \) be a positive real number. Let \( f \in S_{k, \chi}(1) \) with a Fourier expansion as in (1.4). Then \( Pf \) can be written as

\[
P_f(z) = \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) \left( \frac{i}{2\pi} \right)^{k-1-n} \sum_{m+\kappa>0} \frac{a(m)}{(m+\kappa)^{k-1-n}} \times \left( \chi(S)(-1)^n \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \right) \]

\[
  \times \left( \chi(S)(-1)^n \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \right) \]

\[
  \times \left( \chi(S)(-1)^n \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \right) + (-1)^{k-1} \Gamma(2\pi|z|(m+\kappa), k-1-n)z^n \),
\]

where \( \Gamma(a, n) = \int_a^\infty e^{-t}t^{n-1} \, dt \) denotes the incomplete gamma function.

**Proof of Lemma 5.15.** Assume \( 2 < k \in \mathbb{R} \). We see that

\[
F_1(z) = \frac{\chi(S)}{\Gamma} (-1)^{k-1} \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \]

\[
  \times \left( \chi(S)(-1)^n \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \right) \]

and

\[
F_2(z) = \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^{n} z^n \sum_{m+\kappa>0} \frac{a(m)}{(m+\kappa)^{k-1-n}} \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \]

holds. Therefore, by the same computation as in Proposition 5.13 we see that

\[
F_1(z) = \frac{\chi(S)}{\Gamma} \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^{n} \left( \frac{i}{2\pi} \right)^{k-1-n} z^{k-2-n} \times
\]

\[
  \times \sum_{m+\kappa>0} \frac{a(m)}{(m+\kappa)^{k-1-n}} \]

and

\[
F_2(z) = \sum_{n=0}^{\infty} \left( \frac{k-2}{n} \right) (-1)^{k-1} \left( \frac{i}{2\pi} \right)^{k-1-n} z^n \times
\]

\[
  \times \sum_{m+\kappa>0} \frac{a(m)}{(m+\kappa)^{k-1-n}} \Gamma \left( \frac{2\pi}{|z|}(m+\kappa), k-1-n \right) \]

From this we can get the desired result. □

We are ready to prove Theorem 1.5

**Proof of Theorem 1.5.** By Proposition 1.1 we have

\[
\tilde{H}_{1,m}(Pf)(z) = P(H_{1,m}f)(z).
\]
Now we evaluate both sides at $z = 0$. Since $f$ is an eigenform with an eigenvalue $\lambda$,

$$P(H_{1,m}f)(0) = \bar{\lambda} Pf(0).$$

By simple computation we see that

$$Pf(0) = \int_0^{\infty} f(\tau) \tau^{k-2} d\tau = (-i)^{k-1} \frac{\Gamma(k-1)}{(2\pi)^{k-1}} L(f,k-1).$$

On the other hand, by the definition of the $m$th Hecke operator $\tilde{H}_{1,m}$ we obtain

$$P(H_{1,m}f)(0) = \sum_{A \in \mathcal{X}_m} \sum_{l=1}^{L} \chi(m_l) \psi(A) j(m_l, Az)^{k-2} j(A, z)^{k-2} Pf(m_l A0).$$

If we use lemma 5.15 then we see that $Pf(m_l A0) = S_f(m_l A0)$. Therefore, we have

$$(-i)^{k-1} \frac{\Gamma(k-1)}{(2\pi)^{k-1}} \bar{\lambda} L(f,k-1)$$

$$= \sum_{A \in \mathcal{X}_m} \sum_{l=1}^{L} \chi(m_l) \psi(A) j(m_l, Az)^{k-2} j(A, z)^{k-2} S_f(m_l A0).$$

This completes the proof. \qed

References


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