Unimodularity of zeros of period polynomials of Hecke eigenforms

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Abstract

We prove that all the zeros of the full period function of any Hecke eigenform lie on the unit circle \(|z| = 1\).

1. Introduction

Let \(k\) be a positive even integer, and denote by \(S_k\) (respectively, \(M_k\)) the space of cusp forms (respectively, modular forms) of weight \(k\) for the full modular group \(\text{PSL}_2(\mathbb{Z})\). The fundamental domain for the action of the modular group on the complex upper half-plane \(\mathbb{H}\) is given by

\[
\mathcal{F} = \{z \in \mathbb{H} : -\frac{1}{2} < \Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1, \text{ with equality only if } \Re(z) \geq 0\}.
\]

Given the important role of modular forms, and in particular Hecke eigenforms, in number theory, it is quite natural to study their zeros. Let \(Z(f)\) denote the set of zeros of \(f \in M_k\) in \(\mathcal{F}\).

The number of zeros of \(f\) (which equals \(|Z(f)|\) if \(f\) has no repeated zeros) is determined by the valence formula

\[
\text{ord}_\infty(f) + \sum_{Q \in \mathcal{F}} \frac{\text{ord}_Q(f)}{|\text{stab}(Q)|} = \frac{k}{12},
\]

where \(|\text{stab}(Q)|\) denotes the order of the stabilizer group of \(Q\) in \(\text{PSL}_2(\mathbb{Z})\) (which is 2 for \(i\), 3 for \(e^{2\pi i/3}\) and 1 for any other \(Q \in \mathcal{F}\)). Rankin and Swinnerton-Dyer [14]\(^1\) proved that all the zeros of Eisenstein series lie on the arc \(\delta := \{e^{i\theta} : \pi/3 \leq \theta \leq \pi/2\}\), and Kohnen [8] provided explicit formulas for those zeros. On the other hand, it follows from the proof of the Quantum Unique Ergodicity conjecture by Holowinsky and Soundararajan [7] that if \(f_k\) is a sequence of cuspidal Hecke eigenforms (\(f_k\) of weight \(k\)) and \(\Omega\) is a ‘nice’ set, then the elements of \(Z(f_k)\) become equidistributed in \(\mathcal{F}\) in the sense that

\[
\lim_{k \to \infty} \frac{|Z(f_k) \cap \Omega|}{|Z(f_k)|} = \frac{3}{4\pi} \iint_{\Omega} dx\, dy / y^2
\]

(note that the \(\frac{3}{4\pi}\) is the reciprocal of the parabolic area of \(\mathcal{F}\)). Despite that equidistribution result, Gosh and Sarnak [4] also showed that infinitely many zeros of such \(f_k\) lie on the union of the arc \(\delta\) and the lines \(\Re(z) = 0\) and \(\Re(z) = \frac{1}{2}\) (these are the so-called real zeros of \(f_k\), named as such since the \(j\)-invariant is real on the lines and arc described above).

One natural and useful object attached to a modular form \(f(z) = \sum_{n=0}^\infty a_n e^{2\pi inz} \in M_k\) is its Eichler integral defined by

\[
\mathcal{E}_f(z) := \int_z^{i\infty} (f(\tau) - a_0)(\tau - z)^{k-2}\, d\tau = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=1}^\infty \frac{a_n}{n^{k-1}} e^{2\pi inz}.
\]

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\(^1\)Note that the coauthor of Swinnerton-Dyer on that elegant paper of only two pages is not the well-known Robert Rankin, but rather his daughter Fenella (Fenny) Rankin, who was only about 19 years of age when the paper was written.
Although $E_f(z)$ is not a modular form, it nonetheless behaves nicely under modular transformations (in weight $2 - k$) in a way that gives rise to the so-called period function, which is another important object attached to $f$. The period function is given by

$$r_f(z) := E_f(z) - z^{k-2}E_f\left(\frac{-1}{z}\right).$$

(1)

The odd part $r_f^-(z)$ and even part $r_f^+(z)$ of $r_f(z)$ are defined by

$$r_f^+(z) = r_f(z) + r_f(-z).$$

When $f \in S_k$, $r_f(z)$ is in fact a polynomial (of degree $w := k - 2$) which is explicitly given by

$$r_f(z) = \int_{0}^{\infty} f(\tau)(\tau - z)^{k-2}d\tau,$$

(2)

whereas, for $f \in M_k \setminus S_k$, $r_f(z) \in z^{-1}\mathbb{C}[z]$ (see §4 for a more accurate description). It is not hard to see that the coefficients of $r_f(z)$ involve special values of the $L$-function of $f$ (cf. (9) and (13)). Thus, (1) provides a general description connecting Eichler integrals to those $L$-values. From (1), it is clear that any zero $b$ of $r_f(z)$ gives rise to interesting identities involving the Fourier coefficients of $f$ of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{2\pi inb} = b^{k-2} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{-2\pi in/b},$$

(3)

thus providing a pure relation between the Eichler integrals without explicit appearance of the $L$-values at those zeros. It has also been of interest to study the zeros of $r_f^+(z)$, which give rise to identities between Eichler integrals and the $L$-values of a certain parity. This approach has recently been employed in [6] to yield interesting results on the transcendence of Eichler integrals related to Eisenstein series.

Let $V_w$ denote the space of complex polynomials of degree not greater than $w$, with $V^+_w$ (respectively, $V^-_w$) denoting the subspace of odd (respectively, even) polynomials. There is a natural action of $\text{PSL}_2(\mathbb{Z})$ on $\phi \in V_w$ given by

$$\phi \mapsto \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)(z) = (cz + d)^w\phi\left(\frac{az + b}{cz + d}\right).$$

The action preserves each of $V^\pm_w$, and it can be extended to an action of the group ring $\mathbb{Z}[\text{PSL}_2(\mathbb{Z})]$ in the obvious way. Consider the elements

$$S = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \quad \text{and} \quad U = \left(\begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array}\right),$$

which are of orders 2 and 3 in $\text{PSL}_2(\mathbb{Z})$, respectively, and set

$$Y_w := \{ \phi \in V_w : \phi((1 + S) = \phi((1 + U + U^2) = 0\},$$

$$Y^\pm_w := Y_w \cap V^\pm_w.$$ 

It follows from the modular transformation properties of $f \in S_k$ that $r_f(z) \in Y_w$, with $r_f^+ \in Y^+_w$. The Eichler–Shimura theory implies that the map

$$r^- : S_k \longrightarrow W^-,$$

$$f \longmapsto r_f^-(X)$$

is an isomorphism of vector spaces (whereas $r^+(S_k)$ is a codimension 1 subspace of $Y^+_w$). We refer the interested reader to [9, 15] for many more details and extensions. It follows that a cusp form is uniquely defined by its period polynomial (in fact, the odd part would suffice). This gives further evidence that studying the zeros of period polynomials is as natural as studying
the zeros of the cusp forms themselves. Indeed, the following theorem regarding the zeros of the odd period polynomial of Hecke cusp forms was recently proved by Conrey, Farmer and Imamoglu.

**Theorem 1.1** ([3, Theorem 1.1]). If \( f \in S_k \) is a Hecke eigenform, then the odd period polynomial \( r_f(X) \) has simple zeros at 0, \( \pm 2 \) and \( \pm \frac{1}{2} \) and double zeros at \( \pm 1 \). The rest of its zeros are complex numbers on the unit circle.

The main result of the present paper (cf. Theorem 3.4) is that if one considers the full period polynomial \( r_f(X) \) of a Hecke cusp form \( f \), then in fact all of its zeros are on the unit circle. Furthermore, for the Eisenstein series \( G_k \in M_k \setminus S_k \) (normalized so that the coefficient of \( q \) in the Fourier expansion is 1), the period function has the form

\[
r_{G_k}(X) = \frac{P_{G_k}(X)}{X},
\]

where \( P_{G_k}(X) \) is a polynomial of degree \( k - 1 \) (see [15]), and we shall show below (cf. § 4) that work of Lalín and Smyth [11] implies that \( r_{G_k}(X) \) also has all its zeros on the unit circle.

The paper is organized as follows. In Section 2, we recall some definitions and results from [11] that will be useful for our investigation. We also discuss some connections between period polynomials and \( L \)-series. In Section 3, we apply those results to prove our main theorem (Theorem 3.4), as well as an auxiliary result, which is also of independent interest, on the unimodularity of certain polynomials related to the truncation of certain exponential sums (cf. Theorem 3.1). In Section 4, we recall Zagier’s formulas for the period function of Eisenstein series, and relate that to the work of Murty, Smyth and Wang [12] and of Gun, Murty and Rath [6], as well as the results of Lalín and Smyth [11] mentioned above, culminating in a pleasantly uniform result asserting the unimodularity of zeros of period functions for all Hecke eigenforms (cf. Corollary 4.1).

## 2. Definitions and basic tools

A (complex) polynomial \( P(z) = \sum_{i=0}^{d} c_i z^i \) of degree \( d \) is said to be self-inversive if it satisfies

\[
P(z) = \varepsilon z^d \overline{P} \left( \frac{1}{z} \right)
\]

for some constant \( \varepsilon \) (necessarily of modulus 1), where \( \overline{P} := \sum_{i=0}^{d} c_i z^i \) and the bar denotes complex conjugation. Note that the (perhaps more familiar) class of self-reciprocal polynomials is the special case of (4) with \( \varepsilon = 1 \) and \( \overline{P} = P \). Cohn [2] proved that a polynomial \( P(z) \) has all its zeros on the unit circle if and only if it is self-inversive and its derivative \( P'(z) \) has all its zeros in the closed unit disc \( |z| \leq 1 \). Employing a clever yet simple argument, Lalín and Smyth [11] extended Cohn’s result as follows.

**Theorem 2.1** ([11, Theorem 1]). Let \( h(z) \) be a nonzero complex polynomial of degree \( n \) with all its zeros in \( |z| \leq 1 \). Then, for \( d > n \) and any \( \lambda \) with \( |\lambda| = 1 \), the self-inversive polynomial

\[
P^{(\lambda)}(z) = z^{d-n} h(z) + \lambda z^n \overline{h} \left( \frac{1}{z} \right)
\]

has all its zeros on the unit circle.
On inspecting the proof given in [11], it is not hard to see that the condition \( d > n \) could be relaxed to \( d \geq n \). We thus obtain the following slightly stronger version (that would help simplify our proof of Theorem 3.4). The proof is the same as in [11] but we shall include it for completeness.

**Theorem 2.2.** Let \( h(z) \) be a nonzero polynomial of degree \( n \) with all its zeros in \( |z| \leq 1 \). For \( d \geq n \) and \( \lambda \in \mathbb{C} \) set

\[
P^{(\lambda)}(z) = z^{d-n}h(z) + \lambda z^n \bar{h} \left( \frac{1}{z} \right).
\]

(6)

Then, for any \( |\lambda| = 1 \) for which \( P^{(\lambda)} \neq 0 \), \( P^{(\lambda)} \) has all its zeros on the unit circle.

**Proof.** Write \( h^*(z) := z^n \bar{h}(1/z) \), and temporarily assume that all \( n \) zeros of \( h(z) \) are in the open disc \( |z| < 1 \). It follows easily that \( z^{d-n}h(z) \) has all of its \( d \) zeros inside \( |z| < 1 \), whereas \( h^*(z) \) has all of its \( n \) zeros in \( |z| > 1 \). For \( |\lambda| = 1 \), we have \( \bar{z} = 1/z \) and thus

\[
|h^*(z)| = |\bar{h}(\bar{z})| = |\bar{h}(z)| = |h(z)| = |z^{d-n}h(z)|.
\]

Recall that Rouché's theorem states that if \( f \) and \( g \) are functions holomorphic inside and on some closed simple contour \( K \) with \( |g(z)| < |f(z)| \) on \( K \), then \( f \) and \( f + g \) have the same number of zeros inside \( K \). If \( |\lambda| < 1 \), then we set \( f(z) = z^{d-n}h(z) \) and \( g(z) = \lambda h^*(z) \) and we deduce that \( P^{(\lambda)}(z) \) has all of its \( d \) zeros in \( |z| < 1 \). If \( |\lambda| > 1 \), then we can switch the choice of \( f \) and \( g \) to deduce that \( P^{(\lambda)}(z) \) has no zeros in \( |z| < 1 \), and hence all \( d \) of its zeros must be in \( |z| > 1 \). As the zeros of \( P^{(\lambda)}(z) \) are continuous functions of \( \lambda \), we see that, for \( |\lambda| = 1 \), \( P^{(\lambda)} \) must have all its zeros on the unit circle. The result under the weaker assumption that \( h \) has all its zeros in the closed unit disc \( |z| \leq 1 \) follows from continuity of the zeros of \( P^{(\lambda)}(z) \) as functions of the zeros of \( h(z) \).

Given a cusp form \( f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k \) (\( q = e^{2\pi i \tau} \) throughout), let \( L(f, s) = \sum_{n=1}^{\infty} (a_f(n)/n^s) \) be its associated \( L \)-series. The \( L \)-series is well known to converge absolutely for \( s > \frac{k}{2} + 1 \). Setting

\[
L^*(f, s) = \int_0^\infty f(iy) y^s \frac{dy}{y},
\]

we see that \( L^*(f, s) \) is an entire function that satisfies

\[
L(f, s) = (2\pi)^s \frac{L^*(f, s)}{\Gamma(s)},
\]

(7)

thus providing an analytic continuation that can be seen to satisfy the functional equation

\[
\frac{(2\pi)^{k-s}}{\Gamma(k-s)} L(f, s) = j^k \frac{(2\pi)^s}{\Gamma(s)} L(f, k-s).
\]

(8)

Further, using (2) we see that the period polynomial \( r_f(X) \) for a cusp form \( f \) of weight \( k = w + 2 \) can be written as

\[
r_f(X) = -\sum_{n=0}^{w} \frac{w!}{n!} \frac{L(f, w-n+1)}{(2\pi i)^{w-n+1}} X^n = -\frac{w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(f, w-n+1) \frac{(2\pi i X)^n}{n!}.
\]

(9)

For convenience, we shall consider the polynomial with real coefficients

\[
p_f(X) := \sum_{n=0}^{w} L(f, w-n+1) \frac{(2\pi X)^n}{n!} = \left( \frac{2\pi i}{w!} r_f \left( \frac{X}{i} \right) \right).
\]
Using (8), or equivalently the fact that $p_f(1 + S) = 0$, it is easy to see that $p_f(X)$ is self-inversive; indeed we have

$$p_f(X) = i^k X^w p_f \left( \frac{1}{X} \right).$$

Set

$$q_f(X) = \sum_{n=0}^{w-1} L(f, w - n + 1) \frac{(2\pi X)^n}{n!} + \frac{1}{2} L(f, k/2) \frac{(2\pi X)^{w/2}}{(w/2)!} \tag{10}.$$ 

Then, we have

$$i^k p_f(X) = q_f(X) + i^k X^w q_f \left( \frac{1}{X} \right).$$

It is obvious that $r_f(X)$ would have all its zeros on $|z| = 1$ if and only if the same is true for $i^k p_f(X)$. By Theorem 2.2, the latter is true if $q_f(X)$ has all its zeros in $|z| \leq 1$. In Section 3, we will prove the latter, obtaining along the way some similar results for truncations of certain exponential series.

3. Main results and proofs

The idea behind our proof is that, for a normalized (i.e. with leading Fourier coefficient 1) Hecke eigenform $f \in S_k$, the values of $L(f, s)$ for $s$ ‘close’ to $k - 1$ are close to 1, and so the initial part of the period polynomial, namely $q_f(X)$ as in (10) is ‘close’ to the initial part of the series for $e^{2\pi X}$. Before we make that precise (cf. Lemma 3.2), and as a natural step that is easier and also of interest in its own right, we prove that certain self-reciprocal polynomials built from truncations of $e^{2\pi X}$ are eventually unimodular. To that end, set

$$T_m(z) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^n,$$

$$H_m(z) = z^m T_m \left( \frac{1}{z} \right) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^{m-n}$$

and consider

$$P_m^{(\lambda)}(z) := z^m H_m(z) + \lambda T_m(z), \quad \text{with } |\lambda| = 1.$$ 

We will show that the zeros of $P_m^{(\lambda)}(z)$ are unimodular for $m \geq 20$. Note that the assertion is in fact not true for $m$ below that range!

**Theorem 3.1.** For $m \geq 20$, $H_m(z)$ has all of its zeros in $|z| < 1$, and hence $P_m^{(\lambda)}(z)$ has all of its zeros on the unit circle $|z| = 1$.

**Proof.** We start by finding a lower bound for $\min_{|z|=1} |H_m(z)| = \min_{|z|=1} |T_m(z)|$. Write

$$e^{2\pi z} = T_m(z) + R_m(z).$$

By the well-known Taylor inequality, we get, for $|z| = 1$,

$$|R_m(z)| \leq \frac{(2\pi)^{m+1}}{(m+1)!} e^{2\pi} \leq 0.0007513 \quad \text{for } m \geq 25.$$ 

For $z = e^{i\theta}$, we see that

$$|e^{2\pi z}| = e^{2\pi \cos \theta} \geq e^{-2\pi} \geq 0.001867.$$
It thus follows that, for all $|z| = 1$ and $m \geq 25$, we have

$$|T_m(z)| = |e^{2\pi z} - R_m(z)| \geq 0.001867 - 0.0007513 = 0.0011157.$$ 

For $m \geq 25$, we can write $H_m(z) = z^{m-25}H_{25}(z) + g_m(z)$, where

$$g_m(z) := \sum_{n=26}^{m} \frac{(2\pi)^n}{n!}z^{m-n}.$$ 

For $|z| = 1$, we have

$$|g_m(z)| \leq \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} \leq \sum_{n=26}^{\infty} \frac{(2\pi)^n}{n!} = e^{2\pi} - H_{25}(1) \leq 0.000001823 < |H_{25}(z)|.$$ 

It thus follows by Rouché’s theorem that $H_m(z)$ and $z^{m-25}H_{25}(z)$ have the same number of zeros inside the unit circle. Numerical verification using PARI [13] gives that $H_{25}(z)$ has all 25 of its zeros inside the unit circle (the largest of which has modulus less than 0.8); thus, it follows that $H_m(z)$ has all $m$ of its zeros inside the unit circle as well. For $20 \leq m \leq 24$, the result can be verified directly using PARI as well. 

Next, we recall the following estimates from [3].

**Lemma 3.2 ([3, Lemma 2.4]).** Let $f \in S_k$ be a normalized Hecke eigenform and let $L(f,s)$ be its associated $L$-function. Then, for $\Re(s) \geq 3k/4$, we have

$$|L(f,s) - 1| \leq 5^{-k/4}$$

and, for $\Re(s) \geq k/2$, we have

$$L(f,s) \leq 1 + 2\sqrt{k}\log(2k).$$

**Remark 3.3.** The proof of (11) given in [3] seems to contain a minor error that leads to a bound of $4 \times 2^{-k/4}$ rather than $5 \times 2^{-k/4}$. That could be easily remedied by using the estimate (for $k \geq 12$)

$$(\zeta(k/4) + 1)(\zeta(k/4) - 1) \leq (\zeta(3) + 1)(\zeta(k/4) - 1) < 2.5(\zeta(k/4) - 1)$$

in place of the incorrect one $(\zeta^2(k/4) - 1 \leq 2(\zeta(k/4) - 1))$ used in the first formula of that proof.

**Theorem 3.4.** If $f \in S_k$ is a Hecke eigenform, then $r_f(X)$ has all of its zeros on the unit circle.

**Proof.** Note that, for any complex number $\mu$, we have $r_{\mu f}(X) = \mu r_f(X)$; so we might assume, without loss of generality, that $f$ is normalized. By Theorem 2.2, it suffices to prove that all the zeros of $q_f(X)$ are inside the unit circle. Let $m = k/2 - 1$; then, for $|X| = 1$, we have, using Lemma 3.2,

$$|H_m(X) - q_f(X)| \leq \sum_{n=0}^{m-1} |L(f,k-n-1) - 1| \frac{(2\pi)^n}{n!} + \left| 1 - \frac{L(f,k/2)}{2} \right| \frac{(2\pi)^m}{m!}$$

$$\leq \sum_{n=0}^{\lfloor k/4 \rfloor - 1} 5 \times 2^{-k/4} \frac{(2\pi)^n}{n!} + \sum_{n=\lfloor k/4 \rfloor}^{m} (1 + |L(f,k-n-1)|) \frac{(2\pi)^n}{n!}$$

$$\leq 5e^{2\pi}2^{-k/4} + (2 + 2\sqrt{k}\log(2k))R_{\lfloor k/4 \rfloor}(1).$$
Applying Taylor’s inequality as in the proof of Theorem 3.1, we see that
\[ R_{[k/4]} \leq e^{2\pi (2\pi)^{[k/4]}/[k/4]}. \]
It is not hard to show that, for \( k \geq 124 \), we have
\[ (2 + 2\sqrt{k} \log(2k)) e^{2\pi (2\pi)^{[k/4]}/[k/4]} \leq 0.000045 \]
and
\[ 5e^{2\pi} 2^{-k/4} \leq 0.000025. \]
It thus follows that, for \( k \geq 124 \) and \( |X| = 1 \), we have \( |H_m(X) - q_f(X)| < |H_m(X)| \), and it follows from Rouché’s theorem that \( q_f(X) \) has the same number of zeros as \( H_m(X) \) inside the unit circle, namely \( m \) by Theorem 3.1. For cusp forms with \( 12 \leq k \leq 122 \), it can be verified directly (using PARI [13], for instance) that \( q_f(X) \) has all its zeros in \( |z| < 1 \), and thus the result follows for all \( k \).

4. Period polynomials of Eisenstein series

Suppose that \( f = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k \) is a modular form that is not necessarily a cusp form. For \( R(s) \) sufficiently large, the \( L \)-series of \( f \) is defined by \( \sum_{n=1}^{\infty} (a_f(n)/n^s) \). Following Zagier [15], we set
\[ L^*(f, s) = \int_{0}^{\infty} (f(iy) - a_f(0)) y^s dy / y. \]
In the domain of convergence we have \( L^*(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s) \). Furthermore, \( L^*(f, s) \) has a meromorphic continuation to all \( s \in \mathbb{C} \) with its only singularities (if \( a_f(0) \neq 0 \) being simple poles with residues \(-a_f(0)\) and \( i^k a_f(0) \) at \( s = 0 \) and \( s = k \), respectively. It also satisfies the functional equation \( L^*(f, s) = i^k L^*(f, k - s) \). It is thus natural (interpreting factorials as gamma functions as is well known) to extend the formula
\[ r_f(X) = \sum_{n \in \mathbb{Z}} i^{1-n} \binom{w}{n} L^*(f, n + 1) X^{w-n}, \]
which is valid for cusp forms to define the period function \( r_f \) by
\[ r_f(X) = \frac{a_f(0)}{k-1} \frac{X^{k+1}}{X} + \sum_{n=0}^{w} i^{1-n} \binom{w}{n} L^*(f, n + 1) X^{w-n}. \]
As is standard, let \( B_k \) denote the \( k \)th Bernoulli number, and \( \sigma_{k-1}(n) := \sum_{d|n} d^{k-1} \). The Eisenstein series of weight \( k \geq 4 \), normalized so that the coefficient of \( q \) is 1, is uniquely given by the \( q \)-expansion
\[ G_k(\tau) := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \]
It was shown by Zagier [15] that the even and odd parts of the Eisenstein period function are given (up to a constant) by
\[ p^+_{G_k}(X) = X^w - 1, \quad p^-_{G_k}(X) = \sum_{n=-1}^{k-1} \frac{B_{n+1} B_{k-n-1}}{(n+1)! (k-n-1)!} X^n, \]
and that the full period function is given by

$$r_{G_k}(X) = -\frac{w!}{2} \left( p_{G_k}(X) + \frac{\zeta(k-1)}{(2\pi i)^{k-1}} p_{G_k}^+(X) \right). \quad (13)$$

Note that $p_{G_k}(X)$ is indeed an odd function since the coefficient of any even power of $X$ is the product of two odd-indexed Bernoulli numbers, where one of the indices is at least 3 (since $k \geq 4$). It turns out that the appearance of $p_{G_k}$ (and in fact $r_{G_k}$) pre-dates the introduction of period polynomials as they already appear in the following remarkable formula of Ramanujan (see [1, p. 276], for instance), valid for any positive integer $k$ and any $\alpha > 0$, $\beta = \pi/\alpha$:

$$\alpha^{-k} \left( \frac{\zeta(2k+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-(2k+1)}}{e^{2\pi n}} \right) = (-\beta)^{-k} \left( \frac{\zeta(2k+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-(2k+1)}}{e^{2\beta n-1}} \right) - 4k \sum_{j=0}^{k+1} \frac{B_{2j} B_{2k+2-j}}{(2j)! (2k+2-2j)!} \alpha^{k+1-j} (-\beta)^j. \quad (14)$$

This formula led Gun, Murty and Rath [6] to introduce what they called the Ramanujan polynomials given, for $k \geq 1$, by

$$R_{2k-1}(z) = \sum_{j=0}^{k} \frac{B_{2j} B_{2k-2j}}{(2j)! (2k-2j)!} z^{2j}. \quad (15)$$

We clearly see that

$$R_{2k-1}(z) = z p_{G_{2k}}(z).$$

The zeros of $R_{2k-1}(z)$ were studied by Murty, Smyth and Wang [12], where they proved that all except four of the zeros are on the unit circle, and they further showed that those four non-unimodular zeros are all real, with largest absolute value not exceeding $22/10$, and in fact the absolute value approaches 2 as $k \to \infty$. Note that if we denote that largest zero by $a$, then the other non-unimodular zeros are simply $-a, 1/a$ and $-1/a$. This result can be seen as parallel to the one by Conrey, Farmer and Imamoğlu [3, Theorem 1.1].

In [5], Grosswald extended Ramanujan’s formula (14) as follows. Set

$$F_k(z) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k} e^{2\pi i n z}.$$ 

Then, for any $z$, in the upper half-plane $\mathbb{H}$ we have

$$F_{2k-1}(z) - z^{2k-2} F_{2k-1} \left( \frac{1}{z} \right) = -\frac{(2\pi i)^{2k-1}}{(2k-2)!} r_{G_{2k}}(z). \quad (16)$$

It should be noted that formulas such as (16) (and consequently (14), which corresponds to $z = i\alpha$) follow from the general theory of Eichler integrals. Indeed, (16) coupled with the properties of the zeros of $R_{2k-1}$ from [12] were the basis of a number of interesting results on the transcendence of certain Eichler integrals in [6]. Motivated by those results, Lalín and Rogers [10] and Lalín and Smyth [11] studied the zeros of

$$P_k(z) = \frac{(2\pi i)^{2k-1}}{(2k)!} \sum_{j=0}^{k} \frac{2k}{2j} B_{2j} B_{2k-2j} (-z^2)^j + \zeta(2k-1)(z^{2k-1} - 1)z$$

$$= -\frac{(2k-2)!}{2 (2\pi i)^{2k-1}} (i z) r^{G_{2k}}(i z) \quad (17)$$

as well as other self-inversive variations of $R_{2k-1}(z)$ (it should be noted, however, that the definitions given in [10, 11] are combinatorial without explicit reference to period polynomials).
Theorem 8 of [11] establishes that $P_k(z)$ has all its zeros on the unit circle. Combining that result with (17) and Theorem 3.4, we readily obtain the following corollary.

**Corollary 4.1.** If $f \in M_k$ is a Hecke eigenform, then the period function $r_f(z)$ has all its zeros on $|z| = 1$.

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