On the Capacity of Additive White Alpha-Stable Noise Channels

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Abstract—Many communication channels are reasonably modeled to be impaired by additive noise. Recent studies suggest that many of these channels are affected by additive noise that is best explained by alpha-stable statistics. We study in this work such channel models and we characterize the capacity-achieving input distribution for those channels under fractional order moment constraints. We prove that the optimal input is necessarily discrete with a compact support for all such channels.

Interestingly, if the second moment is viewed as a measure of power, even when the channel input is allowed to have infinite second moment, the optimal one is found to have finite power.

I. INTRODUCTION

In modeling the noise effect in a communication channel, it is common to assume that the noise is additive and Gaussian distributed due to the Central Limit Theorem which in layman terms states that, if the noise is due to multiple independent sources, then their cumulative effect is asymptotically Gaussian distributed.

Recent studies however suggest that in some circumstances, the additive noise is better explained by non-Gaussian statistics as is the case when modeling the multiuser interference in a network [1], [2]. This was also observed in the context of modeling the Radio-Frequency interference in embedded wireless laptop transceivers [3]. In both setups, the noise was found to be better modeled as a Symmetric Alpha-Stable (SoS) random variable.

This noise model poses multiple challenges to system designers and we intend to address one of them in this work:

• Despite the fact that the Probability Density Function (PDF) of a SoS random variable was proven to exist and exhibit rather “nice” properties, no closed-form expression is known except in two special cases: the Cauchy and the Gaussian ones.

• Such noise distributions have infinite variance, which implies that the received signal has potentially infinite power and any analysis that is based on a Hilbert space approach is not valid anymore.

The “classical” signal processing techniques used in receiver design are hence not necessarily applicable, let alone optimal.

• The channel capacity of a basic linear channel where the output is simply a noisy version of the input is not known and optimal signaling schemes are not known either. A few attempts were made along this direction and as far as the authors know, only numerical evaluation of some achievable rates have been conducted [4], [5]. In this work, we attempt to partially answer these questions.

Earlier information theoretic studies of linear channels under general additive non-Gaussian noise have been conducted as in [6], [7]. However, typically these studies either require a finite noise variance which is not the case in this model, or impose a peak input power condition, a constraint that we will not impose in this work. It is worth noting as well that, it is not clear whether the SoS noise model fits within the general categories described in [7].

In this paper, we study the additive linear channel model, where a SoS noise variable is added to the input that is generally constrained to satisfy a fractional $r$-th order moment constraint: $E[|X|^r] \leq a$, for any real value of $r > 1$. Clearly, the average power constrained channel is a special case of our setup. More importantly, if $r$ is chosen to be less than 2, then the input is technically allowed to have infinite power similarly to the noise and the received signal: a primary motivation behind this particular choice of the input constraint. Since the received signal is of infinite power, and if a transceiver’s hardware is capable of coping with such a received signal, then it should be able to cope with a similar transmitted one.

The rest of the paper is organized as follows: We first describe the channel model in Section II and derive a lower bound on the output PDF in Section III. Our main results are presented in Section IV, and Section V concludes the paper. Two Appendices I and II establish useful technical lemmas.

II. CHANNEL MODEL

We consider the real linear discrete-time memoryless channel:

$$Y = X + N,$$

where $Y$ is the channel output and $X$ is its input which is subject to the fractional $r$-th order moment constraint: $E[|X|^r] \leq a$ for some real-valued $r > 1$ and $a > 0$. The additive noise variable $N$, which is independent of the input, is assumed to behave statistically as an alpha-stable Random Variable (RV) whose characteristic function is defined by:

$$\Phi(\omega) = e^{i\alpha\omega - |c\omega|^\gamma},$$

where $\alpha \in [1, 2)$, $c > 0$ and $\gamma \in \mathbb{R}$. We note first that, from an information theoretic perspective, assuming the values $\gamma = 0$ and $c = 1$, does not affect the generality of the problem since...
it relates to the equivalent channel where the output is scaled $(Y - \gamma)/c$. Henceforth we assume that the noise, the PDF of which is $p_N(\cdot)$, is a standard SoS RV whose characteristic function is $e^{-|\omega|^\alpha}$.

For additive channels with absolutely continuous noise, the output PDF exists [8]. We denote by $p_Y(y) = p(y; F)^*$ the output density induced by an input probability distribution function $F$. Note that, for $\alpha \in [1, 2)$,

$$p_N(n) \leq \frac{1}{2\pi} \int e^{-|\omega|^\alpha} d\omega = \frac{1}{\pi \alpha} \Gamma \left( \frac{1}{\alpha} \right) < \frac{1}{\sqrt{\pi}} < 1,$$

for all $n \in \mathbb{R}$ and therefore, since $p_N(\cdot)$ is continuous [9, p.11 Th.1.9], less than 1 and has $\mathbb{R}$ as support, so is $p_Y(\cdot)$.

### III. Preliminaries

In order to study the capacity of channel (1) and characterize its achieving input, we derive in this section a lower bound on the output PDF.

It is known that alpha-stable distributions possess heavy tails that decay asymptotically as power laws. More specifically [9, Th.1.12, p.14], let $0 < \epsilon < \frac{1}{2}$, there exists $n_1 > 0$ such that $\forall |n| \geq n_1$,

$$(1 - \epsilon) g(n) < p_N(n) < (1 + \epsilon) g(n), \quad (2)$$

where $g(n) = \frac{2}{\pi} \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(\alpha) |n|^{-(\alpha+1)} = \frac{4k}{|n|^{\alpha+1}}$.

Additionally, alpha-stable distribution functions are known to be analytic on $\mathbb{R}$ whenever $\alpha \in [1, 2)$ [10, p.183 Sec.36]. Whenever $\alpha > 1$ the radius of convergence is $\infty$ and is no less than one if $\alpha = 1$. Hence, if $S_\delta = \{ z \in \mathbb{C} : |\Re(z)| < \delta \}$, $p_N(\cdot)$ is analytically extensible over $S_\delta$ for $\delta$ small enough. Moreover, we prove in Appendix I the following novel bound: there exists $\kappa > 0$ and $n_2 > 0$ such that

$$|p_N(z)| \leq \frac{\kappa}{|\Re(z)|^{\alpha+1}}, \quad \forall z \in S_\delta : |\Re(z)| \geq n_2, \quad (3)$$

which we use to establish Lemma 2 in Appendix II. In the remainder of this paper, we fix $\epsilon$ and a small enough $\delta$, and we define $n_0 = \max\{n_1, n_2\}$ where $n_1$ and $n_2$ are as above.

**Lemma 1.** For any input probability distribution function, the output PDF $p_Y(y)$ of channel (1) is lower bounded by

$$h(y) = \begin{cases} 
-\frac{k_0}{(2u)^{\frac{\alpha}{2}} - y} & y \leq -y_0, \\
-\frac{k_0}{(2u)^{\frac{\alpha}{2}} + y} & y \geq y_0,
\end{cases}$$

where $y_0 = n_0 + (2u)^{\frac{\alpha}{2}}$.

**Proof:** First note that the statement of this lemma does not hold for all inputs in the Gaussian case. The derivations of the bound in the two ranges follow similar steps. We present them hereafter for $y \leq -y_0$:

$$p(y; F) \geq \int_{|x| \leq (2u)^{\frac{\alpha}{2}}} p_N(y - x) dF(x)$$

$$\geq \int_{|x| \leq (2u)^{\frac{\alpha}{2}}} (1 - \epsilon) g(y - x) dF(x) \quad (4)$$

$$\geq \Pr \{ |X| \leq (2u)^{\frac{\alpha}{2}} \} \frac{1}{2} g \left( y - (2u)^{\frac{\alpha}{2}} \right) \quad (5)$$

where we used the lower bound in (2) to obtain (4), and (5) is justified by the fact that $g(y - x)$ is a decreasing function of $x$ on the given interval. The last inequality is an application of Markov’s inequality and the $r$-th moment constraint. ■

### IV. Main Result

In the following we state and prove the main theorem of this paper

**Theorem.** The capacity achieving input of channel (1), whenever $r > 1$ and $\alpha \in [1, 2)$ is compactly supported, discrete with no accumulation point except possibly at zero.

**Proof:** With the sufficient topological and functional requirements satisfied, we proceed as in [8] and we write the Karush-Kuhn-Tucker (KKT) conditions as being necessary and sufficient conditions for the unique optimal input to satisfy. These conditions state that an input random variable $X^*$ with Cumulative Distribution Function $F^*$ achieves the capacity $C$ of an $r$-th moment constrained channel if and only if there exists $\nu \geq 0$ such that

$$\nu(|x|^r - a) + C + H + \int p_N(y - x) \ln p(y; F^*) dy \geq 0, \quad (6)$$

for all $x \in \mathbb{R}$, with equality if $x$ is a point of increase of $F^*$. $H$ is the entropy of the noise and is finite by virtue of equation (2) along with the continuity of $p_N(\cdot)$. Assume that the points of increase of $F^*$ have a positive accumulation point and let

$$s(z) = \nu(z^r - a) + C + H + \int p_N(y - z) \ln p(y; F^*) dy.$$

The function $s(z)$ is analytic on $S_\delta \setminus \mathbb{R}^-$ by virtue of Lemma 2 proven in Appendix II and by using for $z^r$ the principal branch of the logarithm [11, p.85 Prop.1.6.4]. Since $s(z)$ has accumulating zeros on the positive real axis, by the identity theorem [12], $s(\cdot)$ is identically null on $S_\delta \setminus \mathbb{R}^-$. Therefore,

$$\nu(x^r - a) + C + H = -\int p_N(y) \ln p(y - x; F^*) dy, \quad (7)$$

for all $x \in \mathbb{R}^+$. Examining the integral, let

$$t(x) = -\int_{-\infty}^{\infty} p_N(y) \ln p(y - x; F^*) dy = I_1 + I_2 + I_3.$$
where the interval of integration is divided into three subintervals: \((−∞, x − y_0), [x − y_0, x + y_0]\) and \((x + y_0, ∞)\). Using Lemma 1,

\[
I_1 = -\int_{−∞}^{x−y_0} p_N(y) \ln p(y; F^*) dy \\
\leq -\left(\ln k_o \right) \left[1 - \Pr(N > x − y_0)\right] + \\
(\alpha + 1) \int_{x−y_0}^{x−y_0} p_N(y) \ln \left[(2α)^{\frac{1}{α}} x − y\right] dy.
\]

Using the fact that \(\ln x ≤ x^{\frac{1}{α}}\),

\[
\int_{−∞}^{x−y_0} p_N(y) \ln \left[(2α)^{\frac{1}{α}} x − y\right] dy \\
\leq \int_{−∞}^{x−y_0} p_N(y) \left[(2α)^{\frac{1}{α}} x − y\right] \frac{1}{α} dy \\
\leq \mathbb{E}_N \left[(2α)^{\frac{1}{α}} x − N\right]^{\frac{1}{α}} \\
\leq (2α)^{\frac{1}{α}} − N + \mathbb{E}_N \left[(2α)^{\frac{1}{α}} y\right].
\]

We justify (8) by the fact that \(\sum_{n=1}^{∞} X_i^{β} ≤ \sum_{n=1}^{∞} |X_i|^β\) for \(β ≤ 1\). Since \(\mathbb{E}_N \left[(2α)^{\frac{1}{α}} y\right]\) is finite [10, p.179 Sec.35 Th.3], An inspection of equation (8) along with the fact that \(\Pr(N > x − y_0) \sim \frac{4k_o}{α} (x − y_0)^{−α}\) as \(x → ∞\) [9, Th.1.12, p.14] shows that \(I_1 = o(|x|^β)\). Similarly, it can be shown that \(I_3 = o(|x|^β)\). As for \(I_2\),

\[
I_2 = -\int_{x−y_0}^{x−y_0} p_N(y) \ln p(y; F^*) dy \\
\leq \left[F_N(x + y_0) − F_N(x − y_0\right)] \frac{1}{|y| ≤ y_0} \ln \frac{1}{p_N(y)}
\]

where the maximum exists and is positive since \(p_N(y)\) is continuous and \(0 < p_N(y) < 1\). \(I_2\) is also \(o(|x|^β)\) and hence \(t(x) = o(x^β)\) for which equation (7) is impossible unless \(ν = 0\) which is non-sensible since the constraint is biding, and can be formally ruled out [13]. This leads to a contradiction and rules out the assumption of having a positive accumulation point. Similarly, we rule out a negative accumulation point and in conclusion, the set of points of increase of \(F^*\) has no accumulation point except possibly at 0.

**Compact support of the optimal input**

Having ruled out the possibility that an optimal distribution has a set of points of increase with a non-zero accumulation point, we can immediately conclude that this set is countable on \(R\) by virtue of the fact that \(R\) is Lindelöf, and \(X^*\) is discrete. Denoting by \(\{x_i\} ≥ 0\) the points of increase of \(F^*(x)\) and \(p_i = \Pr(X = x_i) > 0\), the output probability density can be lowerbounded by \(p_N(y) > p_i p(y|x_i)\), ∀\(y ∈ R\); \(∀i ≥ 0\).

Examining the last term in the Left-Hand Side (LHS) of the KKT condition (6), it can be proven to be \(o(|x|^β)\) as in the preceding section. Hence, the LHS of (6), \(LHS(x) = ν|y|^β + o(|x|^β)\) diverges to infinity as \(|x|\) goes to infinity for any \(ν > 0\). However, LHS\((x_i) = 0\) for all points of increase \(x_i\). If these were arbitrarily large, then \(ν\) will be zero which is not possible.

**V. Conclusion**

We studied the channel capacity and the capacity-achieving input distributions for channels affected by SoS additive noise.

We proved that the optimal input is discrete and compactly supported for any fractional order input moment constraint larger than one. We note that, even when imposing a fractional lower order moment constraint on the input and allowing it to be potentially of infinite variance, since the optimal input is compactly supported, its variance is nevertheless finite. This indicates that over these types of channels, surprisingly, it does not increase the rates to transmit with infinite power. We acknowledge that measuring the power of a signal via the second moment might not be adequate in this context.

We also note that the methodology developed in this paper may be readily used for Pareto distributions.

Finally, the results may be readily generalized for non-linear channels of the form \(Y = sgnU(X)|X|^α + N\), where \(N\) is a SoS noise random variable, and where the input is subject to \(E[|X|^β] ≤ a, t > s\). An appropriate change of variable as in [14] yields readily a similar result.

**APPENDIX I**

We study in this appendix the rate of decay of standard SoS distributions with \(α ∈ [1, 2]\) on the horizontal strip \(S = \{z ∈ C : |3(z)| < 1\}\). We prove that \(|p_N(z)| = O\left(\frac{1}{|R(z)|^{1 + α}}\right)\) for \(z ∈ S\) as \(|R(z)| → ∞\).

For the case \(α = 1\), \(p_N(z) = \frac{1}{2π} \frac{y}{z}\) which clearly satisfies our assertion. When \(α > 1\), \(p_N(z)\) can be formally extended on \(C\) as:

\[
p_N(z) = \frac{1}{2π} \int_{−∞}^{∞} e^{−izt−|t|^α} dt = \frac{1}{2π} \int_{−∞}^{∞} e^{−izt+yt−|t|^α} dt
\]

\[
= \frac{1}{2π} \sum_{n=0}^{∞} \frac{y^n}{n!} \int_{−∞}^{∞} t^n e^{−izt−|t|^α} dt,
\]

where we wrote \(z = x + iy\), and used Lebesgue’s Dominated Convergence Theorem (DCT) for the interchange in (9). This is justified since

\[
\left|\sum_{n=0}^{∞} \frac{y^n}{n!} t^n e^{−izt−|t|^α}\right| ≤ \sum_{n=0}^{∞} \left|\frac{y^n}{n!} t^n e^{−|t|^α}\right| = e^{|y||t|−|t|^α},
\]

which is integrable for \(α > 1\).

We study next \(J_0(x) = \int_{−∞}^{∞} t^n e^{−izt−|t|^α} dt\). It is known that \(\lim_{x→∞} x^α 1^n J_0(x) = 2Γ(α + 1)\sin \frac{πα}{2}\) [15, p.134].

\footnote{By definition, given three positive real-valued functions \(f(\cdot), g(\cdot)\) and \(h(\cdot)\) we write \(f(z) = O(g(z))\) as \(h(z) → ∞\) if and only if there exist two positive scalars, \(c > 0\) and \(h_o\) such that \(f(z) ≤ cg(z), \forall h(z) ≥ h_o\).}

\footnote{Note that \(J_n(−x) = (−1)^n J_n(x)\) and the \(n\)-th derivative of \(p_N(x)\) is,

\[
p^{(n)}_N(x) = (−x)^n J_n(−x) = \frac{m}{π} J_m(x), \quad n ∈ N.
\]
For $n \geq 1$ and $x > 0$, we have
\[ xJ_n(x) = \left[ \int_0^\infty t^n e^{ixt-x} \, dt + (-1)^n \int_0^\infty t^n e^{-ixt+x} \, dt \right] \]
\[ = i \int_{-\infty}^{\infty} t^{n-1} e^{ixt} \, dt - i \alpha \left[ \int_0^\infty t^{n+\alpha-1} e^{-ixt} \, dt \right] + (-1)^n \int_0^\infty t^{n+\alpha-1} e^{-ixt} \, dt \]
\[ = i n J_{n-1}(x) - \frac{i \alpha}{x^{n+\alpha}} \left[ I_n + (-1)^n T_n \right] \tag{10} \]
and where equation (10) is obtained by integration by parts and regrouping, and where $T_n$ is the complex conjugate of $I_n$: \[ I_n = x^{n+\alpha} \int_0^\infty e^{it^n-x} \, dt = \gamma \int_0^\infty e^{iv^n-\zeta v^n} \, dv, \]
where $\gamma = \frac{1}{n+\alpha}$, $\zeta = x^{-\alpha}$ and the change of variable is $v = (xt)^{n+\alpha}$. Since as $x \to \infty$, $\zeta \to 0^+$,
\[ \lim_{x \to \infty} I_n = \gamma \lim_{\theta \to 0} \int_0^\infty e^{iv^n-\zeta v^n} \, dv \]
\[ = \gamma \lim_{\theta \to 0} \int_0^\infty e^{iv^n(1-\zeta/n)} \, dv \]
\[ = \gamma \lim_{\theta \to 0} \lim_{R \to \infty} \int_1^\infty e^{iz^n} \, dz, \tag{12} \]
where $z = ve^{i\theta}$ for some small positive $\theta$ and $L_1 = \{ z \in \mathbb{C} : z = ve^{i\theta}, 0 < \rho \leq v \leq R \}$. Equation (12) is justified by DCT since the integrand's norm is less than $e^{-\frac{1}{2}v^n}$ which is integrable. Similarly, (13) is justified since the order between the two limits is interchangeable and the integrand in (12) is upperbounded by $e^{-v^n} \sin \gamma \theta$ which is integrable. To evaluate the limit of $\int_{L_1} e^{iz^n} \, dz$, we use contour integration over $\mathcal{C}$ shown in Figure 1. where $C_1$ and $C_2$ are arcs of radius $R$.

On $C_1$, we have:
\[ \lim_{R \to \infty} \left| \int_{C_1} f(z) \, dz \right| \leq \lim_{R \to \infty} \int_0^\varphi iRe^{i\varphi} e^{Re^{i\varphi}} \, d\varphi \]
\[ = \lim_{R \to \infty} \int_0^\varphi Re^{-R^2 \sin(\varphi \theta)} \, d\varphi \]
\[ = 0 \]
where the interchange is valid because $Re^{-R^2 \sin(\varphi \theta)}$ is decreasing as $0 < \gamma \theta \leq \gamma \varphi \leq \frac{\pi}{2}$. Hence, (14) can be shown that $\lim_{R \to \infty} \int_{C_1} f(z) = 0$. It remains to evaluate the integral on $L_2$ where $z = te^{i\pi}$,
\[ \lim_{R \to \infty, \rho \to 0} \int_{L_2} f(z) \, dz \]
\[ = -e^{i \pi} \int_0^\infty e^{-it^2} \, dt = -e^{i \pi} \frac{1}{2} \Gamma \left( \frac{1}{2} \right). \]
In conclusion $\lim_{x \to \infty} I_n = e^{i \frac{\pi}{2} (n+\alpha)} \Gamma(n+\alpha)$, and by (11), we can write for $n \geq 1$
\[ \left| \lim_{x \to \infty} \left[ x^{n+\alpha+1} J_n(x) - i n x^{n+\alpha} J_{n-1}(x) \right] \right| \]
\[ = -i \alpha \Gamma(n+\alpha) \left[ e^{i \frac{\pi}{2} (n+\alpha) + (-1)^n -1} e^{-i \frac{\pi}{2} (n+\alpha)} \right] \tag{14} \]
Since $\lim x^{n+1} J_n(x)$ exists and is equal to $U_0 = 2 \Gamma(\alpha+1) \sin \left( \frac{\pi}{2} \right)$, then $U_n \approx \lim x^{n+1} J_n(x)$ exists for all non-negative $n$ and is equal to, $U_n = i n U_{n-1} + W_n = n! \left[ i^n U_0 + \sum_{k=0}^{n-1} \frac{\alpha^k}{(n-k)!} W_{n-k} \right]$. Furthermore, for $n \geq 0$,
\[ |U_n| \leq n! \left[ |U_0| + \sum_{k=0}^{n-1} \frac{|W_{n-k}|}{(n-k)!} \right] \]
\[ \leq 4 n! \left[ 1 + \sum_{k=0}^{n-1} (n+1-k) \right] = 2 n! \left( n^2 + 3n + 2 \right). \]
The last inequality is justified using the fact that for $\alpha \in (1,2)$, $\Gamma(\alpha + l)$ is increasing for $l \in \mathbb{N}^\ast$.

Now using (9),
\[ \lim_{x \to \infty} x^{\alpha+1} |p_N(x)| \]
\[ = \frac{1}{2 \pi} \lim_{x \to \infty} x^{\alpha+1} \sum_{n=0}^{\infty} \frac{y^n}{n!} \left| \int_0^\infty e^{-it^2} \, dt \right| \]
\[ = \frac{1}{2 \pi} \sum_{n=0}^{\infty} \frac{y^n}{n!} \lim_{x \to \infty} x^{\alpha+1} J_n(-x) \]
\[ \leq \frac{1}{2 \pi} \sum_{n=0}^{\infty} \frac{|y|^n}{n!} \lim_{x \to \infty} x^{n+\alpha+1} |J_n(x)| = \frac{1}{2 \pi} \sum_{n=0}^{\infty} \frac{|y|^n}{n!} |U_n| \]
\[ \leq \frac{1}{\pi} \sum_{n=0}^{\infty} |y|^n \left( n^2 + 3n + 2 \right), \]
which is finite because $|y| < 1$, and where we used the fact that $f(x) = |x|$ is continuous. The interchange in (15) is valid because the end result is finite. In conclusion,
\[ \lim_{x \to \infty} x^{\alpha+1} |p_N(x)| < \infty \] which concludes our proof.
Lemma 2. The function \( i(z; F) : S_{\delta} \rightarrow \mathbb{C} \) defined by:

\[
z \rightarrow i(z; F) = -\int_{-\infty}^{\infty} p_{\text{y}}(y - z) \ln p(y; F) \, dy,
\]

(16)
is analytic over \( S_{\delta} \), for all \( F \) such that \( \int |x|^n \, dF(x) \leq \alpha \).

Proof: To prove this lemma, we will make use of Morera’s theorem:

a) We start first by proving the continuity of \( i(z; F) \) on \( S_{\delta} \). In fact, let \( z_0 \in S_{\delta}, \rho > 0 \), and \( z \in S_{\delta} \) such that \( |z - z_0| \leq \rho \),

\[
\lim_{z \to z_0} i(z; F) = -\lim_{z \to z_0} \int_{-\infty}^{\infty} p_{\text{y}}(y - z) \ln p(y; F) \, dy
\]

(17)

\[
= -\int_{-\infty}^{\infty} p_{\text{y}}(y - z_0) \ln p(y; F) \, dy = i(z_0; F).
\]

Equation (18) is justified by \( p_{\text{y}}(y - z) \) being a continuous function of \( z \) on \( S_{\delta} \), and (17) by DCT. Indeed, in what follows we find an integrable function \( r(y) \) such that,

\[
|p_{\text{y}}(y - z) \ln p(y; F)| = |p_{\text{y}}(y - z) \ln p(y; F) \leq r(y),
\]

for all \( z \in S_{\delta} \) such that \( |z - z_0| \leq \rho \) and for all \( y \in \mathbb{R} \).

Using (3), we upperbound first \( |p_{\text{y}}(y - z)| \). For \( z \in S_{\delta} \) such that \( |z - z_0| \leq \rho \) and \( y \leq -n_0 + |\Re(z_0)| + \rho \), then \( y \leq -n_0 + |\Re(z_0)| + \rho \) and

\[
|p_{\text{y}}(y - z)| \leq \frac{\kappa}{|y - \Re(z)|^{\alpha + 1}} \leq \frac{\kappa}{(n_0 + |\Re(z_0)| + \rho)^{\alpha + 1}}.
\]

Similarly, for \( y \geq (n_0 + |\Re(z_0)| + \rho) \geq (n_0 + |\Re(z_0)| + \rho) \),

\[
|p_{\text{y}}(y - z)| \leq \frac{\kappa}{|y - \Re(z_0)| - \rho} = \frac{\kappa}{|y - \Re(z)|^{\alpha + 1}}.
\]

Since \( p(y; F) \leq 1 \), we upperbound \( -\ln p(y; F) \) using Lemma 1 to obtain

\[
r(y) = \begin{cases} 
(\alpha + 1)\kappa & \ln \left( \frac{(2\alpha)^{\frac{1}{\alpha}} - y}{k_{\alpha + 1}^{\frac{1}{\alpha + 1}}} \right) 
\leq -\psi, \\
-M \ln M_1 & |y| < \psi, \\
(\alpha + 1)\kappa & \ln \left( \frac{(2\alpha)^{\frac{1}{\alpha}} + y}{k_{\alpha + 1}^{\frac{1}{\alpha + 1}}} \right) 
\geq \psi,
\end{cases}
\]

where

\[
\psi = \max \left\{ n_0 + |\Re(z_0)| + \rho, n_0 + (2\alpha)^{\frac{1}{\alpha}} \right\},
\]

\[
M = \max_{|y| \leq \psi} \max_{\zeta \in S_{\delta} \cap |z - \zeta| \leq \rho} |p_{\text{y}}(y - \zeta)|
\]

\[
M_1 = \min_{|y| \leq \psi} p_\rho(y).
\]

Note that \( M \) is finite since the maximization of \( p_{\text{y}}(\cdot) \) is taken over a compact set that is included in \( S_{\delta} \) where it is analytic, and \( 0 < M_1 < 1 \) since \( p_\rho(\cdot) \) is positive and continuous. The fact that \( r(y) \) is integrable concludes the proof of continuity of \( i(z; F) \).

b) To continue the proof of analyticity, we need to integrate \( i(z; F) \) on the boundary \( \partial \Delta \) of a compact triangle \( \Delta \subset S_{\delta} \). We denote by \( |\Delta| \) its perimeter, \( \eta = \min_{z \in \partial \Delta} \Re(z) \), \( \eta = \max_{z \in \partial \Delta} \Re(z) \) and \( \phi = \max\{n_0 + \max\{|n_0|, |\eta|\}, n_0 + (2\alpha)^{\frac{1}{\alpha}}\} \). By similar arguments as above and using the fact that \( p_{\text{y}}(\cdot) \) is analytic on \( S_{\delta} \) one can prove that \( \int_{\partial \Delta} i(z; F) \, dz = 0 \), which in addition to the continuity of \( i(z; F) \) insures its analyticity on \( S_{\delta} \) by applying Morera’s theorem.

Acknowledgments

The authors would like to thank AUB Professor Farouk Abi-Khuzam for helpful discussions on upper-bounding the analytic extension of the alpha-stable density function.

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