The Sum-Capacity of Discrete-Noise Multiple-Access Channels with Single-User Decoding and Identical Codebooks

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Abstract—We consider an additive multiple-access channel model where all users are constrained to use identical codebooks, and where single-user decoding is performed at the receiver.

We study the sum-capacity of the channel for an arbitrarily large, but finite, number of users. For a noiseless $n$-user channel, we construct a signaling scheme that achieves rates per user that are arbitrarily large, proving that the sum-capacity is infinite, whether the users are average and/or peak power limited or not.

We show that this result still holds whenever an arbitrary discrete-noise component is added, provided there exists a positive lower bound on the separation between noise samples. Whenever the noise is of bounded support, the non power-constrained sum-capacity is also proven to be infinite.

The results are valid for an asynchronous multiple-access channel with single-user decoding, as the appropriate channel model is identical to the one studied in this work.

I. INTRODUCTION

Single-user decoding is still commonly used in multi-access communication systems for its attractive simplicity. In addition, using identical codebooks for all users offers significant advantages for system design simplicity, and whenever time-sharing is not an option. This is the case for asynchronous systems where different users start and end transmissions of their codewords at different (and maybe variable) times.

A common ad-hoc analysis for the sum-capacity and rates per user in such systems is based on the starting assumption that the Multi-User Interference (MUI) (in addition to the potential additive noise) is Gaussian distributed. This assumption is commonly made for two main reasons:

1) The capacity region of the Gaussian multiple-access channel when the users are average-power limited is achieved without time-sharing by Gaussian inputs [1] (and hence the MUI is Gaussian distributed.)

2) By the Central Limit Theorem (CLT), whenever the number of users is large enough and their transmission strategies are identical, the MUI of these average-power limited users is expected to be Gaussian distributed in the limit.

Within the context of asynchronous CDMA transmissions, such an analysis leads to the well-known approximate sum-capacity of 0.72 bits per CDMA chip. In [2], Verdu and Shamai studied CDMA transmissions affected by additive Gaussian noise with random spreading. Using Gaussian input statistics (that are optimal for joint processing), the authors considered randomly and independently chosen spreading sequences at the inputs, and they showed that, having a sufficiently large number of equal-power users, the channel throughput under single-user decoding will converge almost surely to 0.72 bits per CDMA chip.

Under a single-user decoding setup, it is however not clear that Gaussian statistics are optimal at the inputs. If one starts by assuming that the MUI is Gaussian distributed, then the optimality of Gaussian inputs is clear. However, the flaw in such an argument lies in the fact that any chosen input statistic defines that of the MUI (even if it approaches Normal statistics) and the problem at hand is not concave. Said differently, it is not clear that the optimization over the input statistic can be separated from its effect on the MUI.

This observation was noted in other works in a noiseless setting. Garba, Bajcsy and Khan [3] considered for example a discrete-time multi-access adder channel model, for $n$ asynchronous CDMA users sending information simultaneously and independently, a model similar to the one adopted in this paper. Single-user decoding was performed at the receiver side and a specific discrete distribution was assigned to the inputs, one that depends on the number of users. It was shown that for the chosen input distribution, the resulting MUI is non-Gaussian and a sum-rate of 1.6544 bits per CDMA chip was achieved.

In this paper, we formulate the problem of finding the sum-capacity of a multiple-access channel when single user decoding is performed and where users are assigned identical codebooks without assuming that the MUI is Gaussian distributed. The statistics of the MUI are therefore dependent on the input statistics and we indirectly account for these statistics whether the inputs are power limited in some form or not. We propose a signaling scheme that is independent of the number of users and that can achieve arbitrarily large (and equal) rates per user, proving that the sum-capacity is infinite.

Finally, note that our model is also appropriate whenever asynchronous transmissions are assumed as in [3] or whenever
multiple transmitter/receiver pairs are operating independently over a common channel and the multi-user interference at any particular receiver is simply treated as noise. A similar (noiseless) setting was adopted in [4], where Gobert considered an uncoordinated multiple-access binary adder channel where each transmitter has a dedicated decoder.

**Problem Definition**

Consider a discrete-time memoryless multiple-access communication channel having \( n \) independent users, where the output \( Y \) of the channel is given by,

\[
Y = \sum_{i=1}^{n} X_i + Z,
\]

where \( Z \) is a discrete-noise random variable, and \( \{X_i\}_{i=1}^{n} \) are the (potentially power-constrained) symbols sent by users \( \{i\}_{i=1}^{n} \) respectively. At the receiver side, single-user decoding is performed, where every user is decoded individually while treating all others as part of the noise.

Since we constrain all users to use the same codebook, a random coding argument establishes that any rate less than \( I(X_1;Y) \) is achievable by user \( i \), where the mutual information is computed according to input probability laws \( p_{X_1}(\cdot) = \cdots = p_{X_n}(\cdot) = p_X(\cdot) \). Note that the channel transition probability is function of the chosen input distribution \( p_X(\cdot) \).

Since all users use the same codebook, each user will clearly achieve the same rate and the sum-capacity \( C \) is

\[
C \geq n I(X_n;Y),
\]

for all possible choices of \( p_X(\cdot) \).

In what follows, we first study the channel when no noise interferes with the inputs (i.e., \( Z = 0 \)) and then generalize the results to the channel when it is corrupted by arbitrary discrete noise.

**II. Preliminaries**

**A. Motivational Examples**

In the examples below, we consider the two-user channel model, \( Y = X_1 + X_2 \), and choose a discrete uniform input with probability mass function \( p_X(x) = \frac{1}{M} \) over an alphabet \( X \subseteq \mathbb{R} \) of size \( M = 3 \).

**Example 1.** Let \( X = \{0, 1, 2\} \), in which case

\[
Y = \{0, 1, 2, 3, 4\},
\]

\[
p_Y(y) = \begin{cases} 2 \quad \text{at each of } \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{2}{9}, \frac{1}{9} \end{cases}.
\]

Computing the entropy of the output \( Y \), we obtain:

\[
H(Y) = - \sum_{y \in \mathcal{Y}} p_Y(y) \log p_Y(y) = 2.197 \text{ bits},
\]

and the achievable rate per user is,

\[
R = I(X_2;Y) = H(Y) - H(Y|X_2) = H(Y) - H(X_1 + X_2|X_2) = H(Y) - H(X_1) = 2.197 - \log 3 = 0.612 \text{ bits/channel use.}
\]

**Example 2.** Now, let \( X = \{0, 1, 3\} \). The channel output \( Y \) is,

\[
\mathcal{Y} = \{0, 1, 2, 3, 4, 6\},
\]

\[
p_Y(y) = \begin{cases} 1 \quad \frac{2}{9}, \frac{1}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9} \end{cases}.
\]

\[
H(Y) = - \sum_{y \in \mathcal{Y}} p_Y(y) \log p_Y(y) = 2.5 \text{ bits},
\]

and the rate per user is

\[
R = I(X_2;Y) = 2.5 - \log 3 = 0.918 \text{ bits/channel use.}
\]

We note from the above examples that a “less structured” spacing between consecutive input alphabet symbols increased the size of the output alphabet, with probabilities more uniformly spread along the output symbols. This resulted in increasing the entropy of the output \( Y \) and consequently the rate per user over the channel. In what follows, we investigate the largest possible size of the output alphabet when the size of the discrete input alphabet is \( M \).

**B. Maximum Size of the Output Alphabet**

Let \( Y = X_1 + X_2 + \cdots + X_n \), and denote the size-\( M \) input alphabet by,

\[
X = \{m_1, m_2, \ldots, m_M\}.
\]

An output alphabet symbol can be written as \( y_a = \sum a_im_i \), where \( a_i \) represents the number of users who sent the input symbol \( m_i \). Note that the channel output depends only on the \( M \)-tuples \( (a_1, \ldots, a_M) \) and that the \( a_i \)'s belong to the set of non-negative integers less than or equal to \( n \) that satisfy \( \sum_{i=1}^{M} a_i = n \).

Denote by \( T \) the set of these \( M \)-tuples \( a \), the number of which is \( \binom{n+M-1}{n} \). Since \( Y \) is the image of \( T \) by the map \( \phi(a) = \sum a_im_i \), then

\[
|\mathcal{Y}| \leq |T| = \binom{n+M-1}{n}.
\]

This upper-bound on the size of the output alphabet is actually met with equality whenever the following condition is satisfied:

**Lemma 1.** Linear independence of the input alphabet symbols \( \{m_1, m_2, \ldots, m_M\} \) over \( \mathbb{Q} \) (or equivalently over \( \mathbb{Z} \)) is a sufficient condition for the map \( \phi(\cdot) \) to be bijective and for the size of the output alphabet to achieve the combinatorial bound (1) with equality.
Proof: Suppose the $m_i$’s are linearly independent over $\mathbb{Q}$, and let $y_a = \sum a_i m_i$ and $y_b = \sum b_i m_i$ be two possible output symbols. If $y_a = y_b$, then $\sum (a_i - b_i) m_i = 0$. Since the $\{m_i\}$’s are linearly independent over $\mathbb{Z}$, then by definition, $a_i = b_i$ for all $i = 1, \cdots, M$.

As a result, different $M$-tuples will necessarily yield different output symbols and the map $\phi$ is thus also injective which implies that, $|\mathcal{Y}| = |\mathcal{T}| = \binom{n+M-1}{M}$.

\section{Sum-Capacity of a Noiseless Channel}

\subsection{The Two-User Channel}

We consider the following discrete-time model of a memoryless two-user multiple-access channel whose output is given by:

$$Y = X_1 + X_2. \quad (2)$$

\textbf{Theorem 1.} Under single-user decoding and whenever the users are constrained to use identical codebooks, any rate per user for channel (2) is achievable, and hence the sum-capacity of the channel is infinite. This is true whether the inputs are power constrained in some form or not.

Proof: Consider a discrete uniform input distribution $p_X(\cdot)$ on the alphabet $\mathcal{X} = \{m_1, m_2, \ldots, m_M\}$, where the $\{m_i\}$’s are chosen to be linearly independent over $\mathbb{Q}$, and $M \geq 2$. Then by virtue of Lemma 1, the size of the output alphabet is

$$|\mathcal{Y}| = \frac{M(M+1)}{2}.$$

If users 1 and 2 send the same symbol at a given time, they can do so in $\binom{M}{2}$ different ways. Symbols being equally probable, the probability of such an event occurring is $\frac{1}{M^2}$. If users 1 and 2 send different symbols at a given time, they can do so in $\binom{M}{2}$ different ways with the probability of such an event occurring being $\frac{2}{M^2}$. We know that the $\binom{M}{1}$ and the $\binom{M}{2}$ possible realizations of the output are disjoint because of the linear independence of the $\{m_i\}$’s (refer to Lemma 1.) Therefore,

$$H(Y) = -\sum p_i \log p_i = -\left( \frac{M}{1} \right) \frac{1}{M^2} \log \frac{1}{M^2} - \left( \frac{M}{2} \right) \frac{2}{M^2} \log \frac{2}{M^2} = 2 \log M - 1 + \frac{1}{M} \text{ bits}.$$

Hence, an achievable rate by user 2 for example is,

$$R = I(X_2;Y) = H(Y) - H(Y|X_2) = H(Y) - H(X_1) = 2 \log M - \frac{M-1}{M} - \log M = \log M - 1 + \frac{1}{M} = \Theta(\log M).$$

Therefore, a transmission rate per user that is logarithmic in $M$ is achievable. Note that $R$ can be made arbitrarily large by taking larger values of $M$. This implies that the sum-capacity of the two-user noiseless channel is infinite.

Finally, note that one can always choose the $\{m_i\}$’s to be linearly independent over any bounded interval, however large $M$ is. This is in fact possible because the degree of the field extension $[\mathbb{R} : \mathbb{Q}]$ is infinite [6]. Therefore, any potential average and/or peak power constraints on the users can be satisfied.

\subsection{The n-User Channel}

We now generalize the setting and the theorem of the previous Section to $n$-users when the discrete-time memoryless channel model is as follows:

$$Y = X_1 + X_2 + \cdots + X_n, \quad (3)$$

and where individual decoding is performed at the channel output.

\textbf{Theorem 2.} Under single-user decoding and whenever the users are constrained to use identical codebooks, any rate per user for channel (3) is achievable, and hence the sum-capacity of the channel is infinite. This is true whether the inputs are power constrained in some form or not.

Proof: Similarly, consider a discrete uniform input over the size-$M$ alphabet, $\mathcal{X} = \{m_1, m_2, \ldots, m_M\}$, where $M \geq n$. Furthermore, assume that the $\{m_i\}$’s are real numbers chosen to be linearly independent over $\mathbb{Q}$. By the result of Lemma 1, the map $\phi : \mathcal{T} \to \mathcal{Y}$ is bijective; the output alphabet symbols are thus uniquely determined by the different $M$-tuples $a$ as discussed in Section II-B. Since the input probability on $\mathcal{X}$ is uniform, the probability of an output symbol associated with a specific $M$-tuple $a$ is

$$p_Y(y_a) = \frac{n!}{a_1! a_2! \cdots a_M!} \frac{1}{M^n}, \quad (4)$$

where the first term corresponds to the number of length-$n$ sequences that have exactly $a_i$ entries equal to $m_i$ for all $i = 1, \cdots, M$. For notational convenience, we will denote this probability term by $\frac{n!}{M^n}$.

Obtaining an exact expression for the achievable rate per user is tedious, but a lower bound may be readily derived:

$$R \geq I(X_n;Y) = H(Y) - H(Y|X_n) = H(X_1 + \cdots + X_n) - H(X_1 + \cdots + X_{n-1}) \geq \left( \frac{M}{n} \right) \frac{n!}{M^n} \log \frac{M^n}{n!} - (1 + \epsilon) \left( \frac{M}{n-1} \right) \frac{(n-1)!}{M^{n-1}} \log \frac{M^{n-1}}{(n-1)!},$$

where we have used the lower and upper bounds derived in Lemma 2 stated hereafter, and where $\epsilon$ is a positive scalar that is $\Theta\left( \frac{1}{M} \right)$.
Therefore,

\[ R \geq \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} \]

\[ - (1 + \epsilon) \left[ \frac{M}{n-1} \right] \log \left( \frac{M^n-1}{(n-1)!} \right) \]

\[ = \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} \]

\[ - M(1 + \epsilon) \left[ \frac{M}{n-1} \right] \log \left( \frac{M^n-1}{(n-1)!} \right) \]

\[ = \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \left( \frac{M^n}{n!} - M(1 + \epsilon) \right) \]

\[ \geq [1 - \delta] \left[ n - \frac{(1 + \epsilon)(n - 1)}{1 - \frac{n-1}{M}} \right] \log M \]

\[ + \frac{1 + \epsilon}{1 - \frac{n-1}{M}} \log(n - 1)! - \log n! \]

where we have used relation ii) in the Appendix to derive the first equality, and where \( \delta \) is a positive scalar, defined in iii) in the Appendix. In conclusion,

\[ R \geq f(n, M) \log M + g(n, M), \]

where

\[ f(n, M) = [1 - \delta] \left[ n - \frac{(1 + \epsilon)(n - 1)}{1 - \frac{n-1}{M}} \right] \]

\[ g(n, M) = [1 - \delta] \left[ \frac{1 + \epsilon}{1 - \frac{n-1}{M}} \log(n - 1)! - \log n! \right] . \]

Note that, for any fixed \( n \), \( f(n, M) \) converges to 1 and \( g(n, M) \) converges to \( \log \frac{1}{M} \) as \( M \) tends to infinity because \( \epsilon \) and \( \delta \) are \( \Theta \left( \frac{1}{M} \right) \). Thus, \( R = \Theta \left( \log \frac{1}{M} \right) = \Theta \left( \log M \right) \) and we have found an input scheme that achieves a transmission rate per user which grows logarithmically with the size of the input alphabet \( M \). Therefore, the sum-capacity of this channel is infinite.

As previously stated, it is always possible to choose the \( \{m_i\} \)'s to be linearly independent over any bounded interval, however large \( M \) is. Therefore, any potential average and/or peak power constraints on the users can be satisfied.

**Lemma 2.** Whenever a uniform discrete input is used over the alphabet \( X = \{m_1, m_2, \ldots, m_M\} \), and whenever the \( \{m_i\} \)'s are real numbers linearly independent over \( \mathbb{Q} \), the entropy of the output of channel (3) when \( M \geq n \geq 2 \) can be upper-bounded and lower-bounded as follows:

\[ H(Y) \geq \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} \]

\[ H(Y) \leq (1 + \epsilon) \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} \]

for some \( \epsilon > 0 \), where \( \epsilon = \Theta \left( \frac{1}{M} \right) \).

**Proof of Lemma:** With the expression of the output probabilities at hand (4),

\[ H(Y) = \sum_{a \in T} \frac{n!}{a^n} \log \frac{a^n}{n!} \]

\[ = \sum_{a_i \leq 1 \forall i} \frac{n!}{a^n} \log \frac{a^n}{n!} + \sum_{\exists i: a_i > 1} \frac{n!}{a^n} \log \frac{a^n}{n!} \]

\[ = \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} + \sum_{\exists i: a_i > 1} \frac{n!}{a^n} \log \frac{a^n}{n!} \]

\[ \geq \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} , \]

because \( -x \log x \) is non-negative in the range \( x \in (0, 1] \). As for the upper bound, it may be readily obtained by noting that

\[ \sum_{\exists i: a_i > 1} \frac{n!}{a^n} \log \frac{a^n}{n!} \leq \sum_{\exists i: a_i > 1} \frac{n!}{M^n} \log \frac{M^n}{n!} \]

\[ = \left( \frac{n + M - 1}{n} \right) \frac{M^n}{n!} \log \frac{M^n}{n!} \]

\[ = \epsilon \left( \frac{M}{n} \right)^n \frac{n!}{M^n} \log \frac{M^n}{n!} , \]

for some \( \epsilon > 0 \) that is \( \Theta \left( \frac{1}{M} \right) \), defined in equality i) in the Appendix. Note that the inequality is valid because \( -x \log x \) is increasing in the range \( x \in (0, 1/2] \), and \( \frac{n!}{2^nM^n} \leq \frac{n!}{M^n} \leq \frac{1}{2} \), whenever \( M \geq n \geq 2 \).

**IV. DISCRETE-NOISE CHANNEL**

Consider now the same multiple-access channel subject to an additive discrete-noise component,

\[ Y = X_1 + X_2 + \cdots + X_n + Z, \]

where \( Z \) is an arbitrary discrete-noise variable. In what follows, we consider two classes of noise distributions, namely,

- one where the possible values of the noise samples have a positive minimum separation, as is the case when the noise is Poisson-distributed,
- another where the noise sample values fall in a bounded interval.

**A. Minimum Separation**

Assume that the minimum separation between noise sample values is lower-bounded by some positive real number \( v \).

**Theorem 3.** Under single-user decoding and whenever the users are constrained to use identical codebooks, any rate per user for channel (5) is achievable, and hence the sum-capacity of the channel is infinite. This is true whether the inputs are power constrained in some form or not.

**Proof:** For such noise distributions, choose the input alphabet \( \{m_1 < \cdots < m_M\} \) such that

\[ n(m_M - m_1) < \nu \]

With the above condition satisfied, the sets of possible output values for each noise sample will be disjoint. That is,
whatever the noise realization is, the decoder will readily be able to uniquely determine it. Once this value is known to the receiver, the problem boils down to the noiseless channel scenario, which has infinite sum-capacity as per the result of Theorem 2.

We note that the construction in this case depends inherently on the number of users via condition (6). As the number of users increases, the input alphabet is more tightly packed.

B. Bounded Support

In this scenario, we assume that the support of \( Z \) is contained in a bounded interval \([a, b] \subseteq \mathbb{R}\).

**Theorem 4.** Under single-user decoding and whenever the users are constrained to use identical codebooks, any rate per user for channel (5) is achievable, and hence the sum-capacity of the channel is infinite.

**Proof:** If the support of the noise is contained in the interval \([a, b] \), the problem can be reduced to the noiseless scenario by choosing the input alphabet \( \{m_1 < \cdots < m_M\} \) such that

\[
\min_{k=2, \ldots, M} \{m_k - m_{k-1}\} > (b - a).
\]

Note that this choice guarantees that distinct output values are separated by at least \((b - a)\). Therefore, whatever realization of the noise occurs, the decoder will know unambiguously which “noiseless” output symbol was meant to be seen had the channel been noiseless. So satisfying the above condition guarantees constructing an input scheme that would result in a robust set of output values, completely noise-immune, and hence transforming the channel into a quasi-noiseless one. ■

Note that the derivation of the theorem did not make use of the fact that the noise is discrete. Indeed, if the additive noise is absolutely continuous but its support included in a bounded interval, then the result of Theorem 4 still applies.

When the noise component \( Z \) has both an unbounded support and no minimum positive separation between noise samples, the capacity results do not necessarily hold. Nonetheless, it is the belief of the authors that a careful construction of the input alphabet will lead in a majority of cases to infinite capacities, a subject yet to be investigated.

V. Conclusion

We have shown that the sum-capacity of the memoryless multiple-access channel under consideration is infinite even when users are constrained to use identical codebooks. This result holds true for the noiseless multiple-access channel and the discrete-noise multiple-access channel whenever we have additional power constraints on the inputs or not, provided that the noise values are separated by at least a positive constant, such as in a Poisson-distributed noise example.

With a large number of users, the chosen input distribution does not yield a Gaussian MUI (since the linear independence of the alphabet symbols over \( \mathbb{C} \) rules out a zero-mean input.) However, with appropriate shifting and scaling of the output, the (shifted) input can be made zero-mean, which yields a MUI for which the CLT applies. Our results indicate that even when the MUI can be approximated by Gaussian statistics, the users’ distribution cannot be studied independently of its effect on the MUI.

The case where the channel is subject to additive white Gaussian noise is fundamentally different from the cases studied in this paper. We expect the sum-capacity and the “good” signaling strategies to be quite different from what was obtained in this work. Indeed, the finiteness of the capacity region under average power constraints imply that the sum-capacity is finite in this case, a subject for future considerations.

APPENDIX A

We present in what follows three identities that were used in various proofs in the paper.

i) \[
\binom{n + M - 1}{n} = (M + n - 1)! / n!(M - 1)!.
\]

ii) \[
\binom{M}{n} n! / M^n = (M(M - 1) \cdots (M - n + 1) / M^n - 1) / M^n.
\]

iii) \[
\binom{M}{n} n! / M^n = M(M - 1) \cdots (M - n + 1) / M^n = 1 - (1 - n + 1).\]

where \( \delta \) is a positive quantity that is \( \Theta(1 / M) \).

ACKNOWLEDGMENTS

The authors would like to thank AUB Professor Kamal Khuri-Makdisi for helpful discussions on various combinatorial bounds.

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