Chapter 7 The Capital Asset Pricing Model

• The capital asset pricing model (CAPM) gives the price of a risky asset within the framework of the mean-variance setting.

• **Market equilibrium**
  
  ➢ Assume that
    
    o All investors in the market are mean-variance optimizers.
    o All investors have the same estimates for the mean and variance and covariance of assets.
    o There is a unique risk-free rate of borrowing and lending available to all investors.
  
  ➢ The one-fund theorem then implies that all investors will purchase the same single fund (portfolio) of risky assets in addition to borrowing or lending at the risk-free rate.
  
  ➢ What must the single risky fund be?
  
  ➢ The CAPM indicates that the single risky fund is the *market portfolio*.
  
  ➢ The market portfolio contains shares of every asset in the market.
  
  ➢ The weight of an asset in the market portfolio is the proportion of that asset’s total capital to the total market capital value. (This is the asset’s capitalization weight.)
Why? – Because of market equilibrium.

- If an asset is not in the market portfolio, then no one buys it. The price of the asset will then be decreased until it is included in the market portfolio and investors buy it.
- If an asset’s demand exceeds its outstanding number of shares, then the price of the asset will be increased until demand meets supply.

- **The capital market line**
  
  - Let $M$ be the market portfolio. The efficient set is then the line in the $r - \sigma$ plane connecting $M$ and the risk free asset.

  \[ r = r_f + \frac{r_M - r_f}{\sigma_M} \sigma. \]

  - This line is called the *capital market line*.
  - It shows the relation between expected return and risk. The higher the risk the higher the expected return.
  - The line equations is

  $r = r_f + \frac{r_M - r_f}{\sigma_M} \sigma.$

  - The slope $K = (r_M - r_f) / \sigma_M$ is called the price of risk.
• The pricing model
  ➢ Under CAPM, the expected return of asset \( i \) satisfies
    \[
    \overline{r}_i - r_f = \beta_i (\overline{r}_M - r_f)
    \]
    (CAPM formula)
    where \( \beta_i = \sigma_{iM} / \sigma^2_M \) is the beta of asset \( i \).
  ➢ This relationship can be shown by exploiting the tangency conditions between the curve representing the two-asset portfolio composed of asset \( i \) and \( M \) and the market line.
  ➢ The value \( \overline{r}_i - r_f \) is called the *expected excess rate of return*.
  ➢ The CAPM formula states that it’s the beta of the asset (i.e., the covariance with the market portfolio) that determines the asset expected return.
  ➢ The CAPM changes the concept of risk of an asset from standard deviation (\( \sigma \)) to covariance (\( \beta \)).
  ➢ The logic is that the risk of an asset which is not correlated with the market can be diversified away.
The concept of beta is well established in the financial community. It indicates the correlation between a company’s performance and market conditions.

Average betas in the U.S.

<table>
<thead>
<tr>
<th>Industry Name</th>
<th>Average Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-Commerce</td>
<td>3.04</td>
</tr>
<tr>
<td>Semiconductor</td>
<td>2.97</td>
</tr>
<tr>
<td>Internet</td>
<td>2.78</td>
</tr>
<tr>
<td>Telecom. Equipment</td>
<td>2.61</td>
</tr>
<tr>
<td>Wireless Networking</td>
<td>2.60</td>
</tr>
<tr>
<td>Power</td>
<td>2.23</td>
</tr>
<tr>
<td>Drug</td>
<td>1.59</td>
</tr>
<tr>
<td>Advertising</td>
<td>1.56</td>
</tr>
<tr>
<td>Entertainment</td>
<td>1.47</td>
</tr>
<tr>
<td>Air Transport</td>
<td>1.40</td>
</tr>
<tr>
<td>Healthcare Information</td>
<td>1.38</td>
</tr>
<tr>
<td>Securities Brokerage</td>
<td>1.36</td>
</tr>
<tr>
<td>Educational Services</td>
<td>1.09</td>
</tr>
<tr>
<td>Recreation</td>
<td>1.08</td>
</tr>
<tr>
<td>Manuf. Housing/RV</td>
<td>1.08</td>
</tr>
<tr>
<td>Medical Supplies</td>
<td>1.04</td>
</tr>
<tr>
<td>Chemical (Basic)</td>
<td>1.03</td>
</tr>
<tr>
<td>Shoe</td>
<td>1.02</td>
</tr>
<tr>
<td>Retail Store</td>
<td>0.99</td>
</tr>
<tr>
<td>Tobacco</td>
<td>0.66</td>
</tr>
<tr>
<td>Water Utility</td>
<td>0.64</td>
</tr>
<tr>
<td>Food Processing</td>
<td>0.61</td>
</tr>
<tr>
<td>Beverage (Soft Drink)</td>
<td>0.61</td>
</tr>
<tr>
<td>Food Wholesalers</td>
<td>0.60</td>
</tr>
<tr>
<td>Bank</td>
<td>0.55</td>
</tr>
</tbody>
</table>

(Source: [http://pages.stern.nyu.edu/~ADAMODAR/New_Home_Page/data.html](http://pages.stern.nyu.edu/~ADAMODAR/New_Home_Page/data.html))

For a portfolio of \( n \) assets with weights \( w_1, w_2, \ldots, w_n \), the beta is

\[
\beta = \frac{\text{Cov}(\sum_{i=1}^{n} w_i r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \text{Cov}(r_i, r_M) / \sigma_M^2 = \sum_{i=1}^{n} w_i \beta_i.
\]
• **The security market line**

  ➢ The CAPM formula can be expressed graphically as a linear relationship between expected return and covariance or beta.

  ![Graph of Security Market Line](image)

  ➢ The line in the graphs is termed the security market line. Any asset should fall on this line.

• **Systematic risk**

  ➢ The CAPM formula suggest the following probabilistic structure of an asset rate of return:

  \[
  r_i = r_f + \beta_i (r_M - r_f) + \varepsilon_i, \quad (1)
  \]

  where \( \varepsilon_i \) is a random variable.

  ➢ By taking expectation of both sides of (1), it follows that

  \[
  \bar{r}_i = \bar{r}_i + E[\varepsilon_i], \quad \text{which implies that } E[\varepsilon_i] = 0.
  \]

  ➢ Taking the covariance with \( r_M \) of both sides of (1) implies

  \[
  \text{Cov}(r_i, r_M) = \beta_i \text{Cov}(r_M, r_M) + \text{Cov}(\varepsilon_i, r_M)
  \]

  \[
  \Rightarrow \sigma_{iM} = (\sigma_{iM} / \sigma_M^2)\sigma_M^2 + \text{Cov}(\varepsilon_i, r_M) \Rightarrow \text{Cov}(\varepsilon_i, r_M) = 0.
  \]
Now, taking the variance of both sides implies of (1) implies
\[ \sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{Var}[\varepsilon_i] . \]

That is, the risk of asset \( i \) is composed of two terms: (i) The *systematic risk*, \( \beta_i^2 \sigma_M^2 \), and (ii) the *nonsystematic* or *specific risk* \( \text{Var}[\varepsilon_i] \).

The systematic risk is the risk associated with the market as a whole and cannot be diversified.

The specific risk is uncorrelated with the market and can be diversified.

The systematic risk is the most important.

The nonsystematic risk of an asset on the capital market line is zero because the standard deviation for this asset is
\[ \sigma = \frac{\bar{r} - r_f}{\bar{r}_M - r_f} \sigma_M = \beta \sigma_M . \]

Then, the nonsystematic risk can be represented graphically as the horizontal distance to the capital market line.
• **Investment Implications**
  - Can CAPM help with investment decisions?
  - CAPM recommends that investors should invest in the market portfolio.
  - That is, an investor should purchase a little bit of each available asset without solving a Markowitz model.
  - Mutual funds match the market portfolio closely by investing in *index funds* (e.g. SP 500 index).
  - So, if you believe in CAPM, just invest in an index fund.
  - But an investor may believe he has superior mean-variance estimates and may engage in solving a Markowitz model.
  - However, accurate mean-variance estimates are difficult to obtain.
  - When constructing a portfolio, the recommendation is to start with the CAPM market portfolio and alter it systematically without solving the Markowitz model from scratch.
  - CAPM can also be useful in assigning reasonable prices for assets that do not have well-established prices.

• **Performance Evaluation**
  - The CAPM theory can be used to evaluate the performance of investment portfolios by comparing them to assets on the capital market line (see Example 7.4, text).
Example 7.4: Evaluating ABC fund. Use S&P index as the market fund and T-bills as the risk free asset.

<table>
<thead>
<tr>
<th>Year</th>
<th>ABC</th>
<th>S&amp;P</th>
<th>T-bills</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7</td>
<td>7.5</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>20</td>
<td>7.7</td>
</tr>
<tr>
<td>4</td>
<td>-8</td>
<td>-2</td>
<td>7.5</td>
</tr>
<tr>
<td>5</td>
<td>23</td>
<td>12</td>
<td>8.5</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
<td>23</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>17</td>
<td>7.3</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>.9</td>
<td>.5</td>
<td>7.5</td>
</tr>
<tr>
<td>10</td>
<td>19</td>
<td>16</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Avg.</th>
<th>13.00</th>
<th>12.00</th>
<th>7.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std. Dev.</td>
<td>12.39</td>
<td>9.43</td>
<td>0.47</td>
</tr>
<tr>
<td>Geom. Mean</td>
<td>12.34</td>
<td>11.63</td>
<td>7.50</td>
</tr>
</tbody>
</table>

\[
\hat{\beta} = -0.768. 
\]

From this data, we evaluate \textit{Jensen's index} as

\[
J = \hat{r} - r_f - \beta(\hat{r}_M - r_f) = 13 - 7.6 - 1.20375 \times (12 - 7.6) = 0.1035 > 0. 
\]

So, ABC is located above the security market line suggesting that ABC performs better than CAPM prediction.

Given the limited data set used here. This suggests that ABC is a good asset but does not guarantee that ABC is efficient.
To check whether ABC is efficient we find *Sharpe’s index*, $S$, for both ABC and the market fund.

For ABC, $S_{ABC} = (\hat{\mu} - r_f) / s = (13 - 7.6) / 12.4 = 0.43548$. For the market portfolio, $S_M = (\hat{\mu}_M - r_f) / s_M = (12 - 7.6) / 9.4 = 0.46808$.

Sharpe’s index of an asset gives the slope of the line connecting the risk-free asset and the asset in the $\bar{\mu} - \sigma$ plane.

We conclude that ABS is not efficient on its own. But it is a good candidate to be combined with other assets in a portfolio and achieve efficiency.
• **CAPM as a pricing formula**

- The CAPM formula, \( \bar{r} - r_f = \beta(\bar{r}_M - r_f) \), does not give the price of an asset explicitly.
- To get a formula for the price, recall that \( r = (Q - P) / P \), where \( Q \) is the asset value after one time period and \( P \) is the asset price. Then, \( \bar{r} = (\bar{Q} - P) / P \), where \( \bar{Q} = E[Q] \).
- Then, \( (\bar{Q} - P) / P - r_f = \beta(\bar{r}_M - r_f) \Rightarrow \bar{Q} - P = P(r_f + \beta(\bar{r}_M - r_f)) \).
- This implies the pricing form of CAPM

\[
P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}.
\]
- CAPM pricing may be seen as discounting the expected return of an asset at the risk-adjusted rate \( r_f + \beta(\bar{r}_M - r_f) \).
- One concern is whether the CAPM pricing formula is *linear* in the sense that the price of the sum of two assets is equal to the sum of the two assets prices.
- The linearity property is not directly clear from the formula,

\[
P_1 = \frac{\bar{Q}_1}{1 + r_f + \beta_1(\bar{r}_M - r_f)}, \quad P_2 = \frac{\bar{Q}_2}{1 + r_f + \beta_2(\bar{r}_M - r_f)} \Rightarrow P_1 + P_2 = \frac{\bar{Q}_1 + \bar{Q}_2}{1 + r_f + \beta_1 + \beta_2(\bar{r}_M - r_f)}
\]

- However, it can be shown that linearity holds by working with an alternate form of the formula. Note that

\[
\beta = \frac{\text{Cov}(r, r_M)}{\sigma_M^2} = \frac{\text{Cov}(Q / P - 1, r_M^2)}{\sigma_M^2} = \frac{\text{Cov}(Q, r_M)}{P \sigma_M^2}
\]

- Then, \( P = \frac{\bar{Q}}{1 + r_f + \left[ \frac{\text{Cov}(Q, r_M)}{P \sigma_M^2} \right](\bar{r}_M - r_f)} \).
This leads to the CAPM certainty equivalent form

\[
P = \frac{1}{1+r_f} \left[ \bar{Q} - \frac{\text{Cov}(Q, r_M)(\bar{r}_M - r_f)}{\sigma_M^2} \right].
\]

This formula is indeed linear (why?).

The term \[
\left[ \bar{Q} - \frac{\text{Cov}(Q, r_M)(\bar{r}_M - r_f)}{\sigma_M^2} \right]
\]
is called the certainty equivalent of the asset return \( Q \) since the risk-free (certain) interest rate is applied to this term to get the price.

It is important that linearity holds, because otherwise there are arbitrage opportunities.

- **Project Choice**

  The net present value (NPV) of a project requiring an initial outlay of \( P \) and returning an uncertain amount \( Q \) after one year can be defined based on CAPM as

  \[
  -P + \frac{1}{1+r_f} \left[ \bar{Q} - \frac{\text{Cov}(Q, r_M)(\bar{r}_M - r_f)}{\sigma_M^2} \right].
  \]

  A firm can then select between several project based on their NPV as in the deterministic case.

  Investors in the firm following the CAPM mode would like the firm to operate as to “expand” the efficient frontier (i.e., push the efficient frontier upward and leftward leading to higher returns with low risk).

  The **harmony theorem** states that expanding the efficient frontier and maximizing NPV are equivalent criteria.