Simulating Stock Prices

• The geometric Brownian motion stock price model

➢ Recall that a rv $Y$ is said to be lognormal if $X = \ln(Y)$ is a normal random variable.

➢ Alternatively, $Y$ is a lognormal rv if $Y = e^X$, where $X$ is a normal rv.

➢ If $\nu$ and $\sigma$, are the mean and standard deviation of $X$, the mean and variance of $Y$ are given by

$$
E[Y] = e^{\nu + \sigma^2/2}, \quad \text{var}[Y] = e^{2\nu + \sigma^2} (e^{\sigma^2} - 1).
$$

➢ Note that $E[Y] \neq e^{E[X]} = e^\nu$ although $Y = e^X$.

➢ A popular stock price model based on the lognormal distribution is the geometric Brownian motion model, which relates the stock prices at time 0, $S_0$, and time $t > 0$, $S_t$ by the following relation:

$$
\ln(S_t) = \ln(S_0) + (\mu - \sigma^2 / 2)t + \sigma z(t),
$$

where, $\mu$ and $\sigma > 0$ are constants and $z(t)$ is a normal rv with mean 0 and variance $t$.\(^1\)

➢ It follows that $\ln(S_t / S_0)$ is a normal random variable with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.

\(^1\) $z(t)$ is called a Brownian motion.
That is, \( S_t / S_0 \) is a lognormal rv with mean and variance
\[
E[S_t / S_0] = e^{\mu t - \frac{\sigma^2 t}{2} + \frac{\sigma^2 t}{2}} = e^{\mu t},
\]
\[
\text{var}[S_t / S_0] = e^{2(\mu t - \frac{\sigma^2 t}{2}) + \sigma^2 t} (e^{\sigma^2 t} - 1) = e^{2 \mu t} (e^{\sigma^2 t} - 1).
\]

Note that \( \mu \) can be seen as the stock rate of return assuming continuous compounding. So \( \mu \) is called the expected return of the stock.

In addition, \( \sigma \) measures the variability of the stock price. So \( \sigma \) is called the volatility of the stock price.

Typical values for these parameters are \( \mu = 13\% \) and \( \sigma = 15\% \) when time \( t \) is measured in years.

The main idea behind the geometric Brownian motion model is that the probability of a certain percentage change in the stock price within a time \( t \) is the same at all times.

This is a memoryless or Markovian behavior indicating that past stock values won’t help in predicting future values.

In addition, the expected value and variance of the stock price typically follow an increasing trend.
Simulating geometric Brownian motion stock prices

- The key idea for simulating a stock price is that $\ln(S_t / S_0)$ is normally distributed with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.
- An algorithm for simulating the stock price at a time $t > 0$, given that current stock price (at $t = 0$) is $S_0$ is as follows.
  1. Generate $Z \sim N(0,1)$
  2. Set $\mu_t = (\mu - \sigma^2/2)t$ and $\sigma_t = \sigma t^{0.5}$.
  3. Set $S_t = S_0 e^{\mu_t + \sigma_t Z}$
- In practice the expected return, $\mu$, is too difficult to estimate accurately, while the volatility $\sigma$ can be estimated reasonably well from historical data.
For estimating $\mu$ one is better off making a subjective estimate or a probability distribution.

Estimating $\sigma$ can be made based on historical data. However, an implied volatility approach is often used.

The idea of implied volatility is to find $\sigma$ based on the market prices of certain financial instruments.

Among the widely used instruments for this purpose are European stock options.

- **European options and the Black–Scholes model**
  
  A *European call option* is a financial instrument that gives its holder the right, but not the obligation, to *buy* one (or more) share(s) of stock price for a *strike price* $K$ per share at a *maturity time* $T$ in the future.

  The buyer of the option pays a price or a *premium*, $C$, in exchange for it.

  Obviously, a European call option is beneficial only when the stock price at time $T$ exceeds $K+C$.

  A *European put option* is a financial instrument that gives its holder the right, but not the obligation, to *sell* one (or more) share(s) of stock price for a *strike price* $K$ per share at a *maturity time* $T$ in the future.

  The buyer of the option pays a premium, $P$, in exchange for it.
Assuming a geometric Brownian motion stock model, Black and Scholes (1973) derived a key result given $C$ (or $P$) as a function of $T$, $K$, $\mu$, $\sigma$, and the interest rate $r$ assuming continuous compounding.

The Black-Scholes formula for call and put options is given by the following theorem.

**Theorem** The price at time 0 of a European call and put options with strike price $K$ and maturity $T$ on an underlying stock with volatility $\sigma$ are

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where $S_0$ is the stock price at time 0,

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

and $N(x) = \int_{-\infty}^{x} e^{-x^2 / 2} \sqrt{2\pi} \, dx$ is the standard Normal cdf.

The implied volatility, $\hat{\sigma}$, of a stock is the value of $\sigma$ which make the option price specified by the Black-Scholes formula equal to market prices of the options listed in financial newspapers and websites.

Finding $\hat{\sigma}$ is done numerically.
• **Example 1.**
  - On June 30, 1998 Dell stock sold for $94. A European put with a strike price of $80 expiring on November 22, 1998 was selling for $5.25. The current 90 day T-Bill (bond) rate is 5.5%. What is the implied volatility for Dell?
  - Solution: See Excel file Dell_stock.xls.

• **Example 2 (model with random $\mu$).**
  - Simulate the daily Dell stock in Example 1 between July 1, 1998 till the end of 1998. The expected stock return is believed to equally likely take on values 10%, 20%, 30%, and 40%.
  - Solution: See Excel file Dell_stock.xls.

• **Example 3 (model with $\mu$ based on analyst forecast).**
  - Simulate the daily Dell stock in Example 1 between July 1, 1998 till the end of 1998. Analysts consensus view is that Dell stock will be selling for $110 on 1/1/1999.
  - Here we need to solve for $\mu$ that makes the expected stock price equal to $120 on 1/1/1999. Recall that the expected stock price at time $t$ is $E[S_t] = S_0e^{\mu t}$. 
Setting \( t = 0 \), at 06/30/1998, then \( t = 185 \) on 1/1/1999.

Then, we can solve for \( \mu \) as follows.

\[
110 = 94e^{\mu(185/365)} \Rightarrow \ln(110) = \ln(94) + 0.5068\mu \Rightarrow \mu = 0.31.
\]

See Excel file Dell_stock.xls for the complete simulation.

**Example 4 (Strong Buy/Strong Sell Consensus).**

On July 10, 2000, Business Week reported the results of a study that estimated how well analysts’ consensus of Strong Buy (1) to Strong Sell (5) forecast annual return on a stock. They found the following predictions for annual returns (relative to the market, assessed usually via an index fund).

<table>
<thead>
<tr>
<th>Rating</th>
<th>Average Excess Return over Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong Buy = 1</td>
<td>+4.5%</td>
</tr>
<tr>
<td>Buy = 2</td>
<td>+3.5%</td>
</tr>
<tr>
<td>Hold = 3</td>
<td>+0.5%</td>
</tr>
<tr>
<td>Sell = 4</td>
<td>−1.0%</td>
</tr>
<tr>
<td>Strong Sell = 5</td>
<td>−8.5%</td>
</tr>
</tbody>
</table>

Suppose that Dell stock in Example 1 got a rating of 1.6, and the market return is equally likely to be −10%, −5%, 0%, 5%, 10%, 15%. Simulate the daily Dell stock between July 1, 1998 and the end of 1998.
First, we find the average excess return of Dell over market by interpolating in the Business Week table as $0.4 \times 4.5 + 0.6 \times 3.5 = +3.9\%$.

Then, $\mu$ is set as 3.9% plus the simulated market return.

See Excel file Dell_stock.xls for the complete simulation.

- **Generating Stock Prices by Bootstrapping.**
  - As an alternative approach to geometric Brownian motion, we use bootstrapping to simulate future stock prices based on a sample of stock history.
  - The idea behind bootstrapping is assuming that every past value is equally likely to occur in the future.

- **Example 5 (Bootstrapping).**
  - Simulate the first 3 months of 1999 of the Dell stock price in Example 1 based on data from the last three months of 1998 by bootstrapping.
  - See Excel file Dell_stock.xls for the complete simulation.
• Pricing Options via Simulation

- This is useful for European-style options (such as average and other path-dependent options).
- The arbitrage-free, risk-neutral, price at time $t$ of such options with expiration time $T$ is

$$f(S,t) = e^{-r(T-t)} \hat{E}[f(S,T)],$$

where $\hat{E}$ is the expectation with respect to the Ito process

$$dS(T) = rS(T)dt + \sigma S(T)d\hat{\zeta}.$$  

- Then, the price of the security can be determined via simulation as

$$\hat{f}(S,t) = \frac{1}{n} \sum_{i=1}^{n} e^{-r(T-t)} f(S^i_T, T),$$

where the superscript denotes simulation experiment $i = 1, \ldots, n$.

- For example, for a European call option

$$f(S^i_T, T) = \max(0, S^i_T - K).$$

- Then, one simulates $n$ values for $S^i_T$, as discussed above, and finds the option price.

• Example 6 (Pricing Dell put).

- Consider the same data as Example 1. However, assume now that the volatility is given, and equals 53.26%, and it is required to find the price of the put option.
In this case,
\[ f(S^i_T, T) = \max(0, K - S^i_T) \]

See Excel file Dell_stock.xls for the complete simulation, using @Risk.

**Example 7 (Pricing Dell put).**
- Same as Example 6, but it is required to find the price of the Dell call option with the same strike price and maturity.
- In this case,
\[ f(S^i_T, T) = \max(0, S^i_T - K) \]

See Excel file Dell_stock.xls for the complete simulation, using @Risk.

**Example 7 (Pricing Dell Asian call option).**
- Same data as Example 7, but now the payoff depends on the average of the Dell weekly prices between June 30, 1998 and November 22, 1998.
- In this case,
\[ f(S^i_T, T) = \max(0, \bar{S}_T - K), \]
where \( \bar{S}_T = \frac{\sum_{j=1}^{n\text{weeks}} S_{\text{end of week } j}}{n\text{weeks}} \).

See Excel file Dell_stock.xls for the complete simulation, using @Risk.