Chapter 4 The Term Structure of Interest Rate

- The yield curve
  - Long bonds tend to offer higher yields than short bonds of the same quality.
  - The yield curve display yield as a function of time to maturity.
  - The yield is constructed based on yields of available bonds of a given quality class.
  - A rising yield curve is *normally shaped*. This occurs most often.
  - If long bonds happen to have lower yields than short bonds then the result is an *inverted yield curve*.

(Source: http://en.wikipedia.org/wiki/Yield_curve)
When studying a particular bond it is useful to place it as a point in the plot of the yield curve.

- **The term structure**
  - Term structure theory is based on the observation that interest rate depends on the length of time the money is held.

- **The spot rates**
  - *Spot rates* are the basic interest rates defining the term structure.
The spot rate $s_t$ is the interest rate charged for money held from present till year $t$.

For example, a 1-year deposit will grow by a factor of $(1+s_1)$. A 2-year deposit will grow by a factor of $(1+s_2)^2$.

In general, a $t$-year investment grows by a factor of $(1+s_t)^t$.

Compounding rules applies to spot rates. For example, under a compounding of $m$ times per year, a $t$-year deposit will grow by a factor of $(1+s_t/m)^{mt}$.

Under continuous compounding a $t$-year deposit will grow by a factor of $e^{s_t t}$.

Discount factors and present values can then be determined in the usual way.

For example, with yearly compounding, the present value of a cash flow stream $x = (x_0, x_1, \ldots, x_n)$ is

$$PV = \sum_{k=0}^{n} d_k x_k,$$

where $d_k = 1/(1+s_k)^k$.

- **Spot rates curve**

  Spot rates can be determined from the yields of zero-coupon.
If not enough, zero-coupon bonds are available (especially long ones), the spot rate curve can be determined from the prices of coupon-bearing bonds.

For example, suppose you have a 1-year zero-coupon bond and a 2-year bond paying a coupon $C_2$ every year. The yield of the first coupon ($P_1/F_1$) gives the spot rate $s_1$.

Then, the spot rate $s_2$ can be determined from the equation

$$P_2 = \frac{C_2}{1 + s_1} + \frac{C_2 + F_2}{(1 + s_2)^2}.$$

Spot rates can also be found by “subtraction” of two bonds with different coupons to construct a zero-coupon bond.

(Examples 4.1 and 4.3)

- **Forward rates**

Forward rates are interest rates for money to be borrowed between two future dates, under terms agreed upon today.

E.g., suppose there are two ways of investing $1 for 2 years

(i) Deposit it in a 2-year bank account where it will grow to $(1+s_2)^2$ at the end of the two years.

(ii) Deposit it in a 1-year bank account where it will grow to $(1+s_1)$ at the end of the first year, and then deposit the proceeds for one more year at a rate $f_{1,2}$ (yielding $(1+s_1)(1+f_{1,2})$ at the end of the two years).

In this case, $f_{1,2}$ is the forward rate between years 1 and 2.
Invoking the comparison principle implies that
\[(1 + s_2)^2 = (1 + s_1)(1 + f_{1,2}) \Rightarrow f_{1,2} = \frac{(1 + s_2)^2}{1 + s_1} - 1.\]

The use of the comparison principle can be justified through an \textit{arbitrage} argument.

Arbitrage is earning money without investing anything.

If \((1 + s_2)^2 < (1 + s_1)(1 + f_{1,2})\), then one can borrow $1 for two years and invest it according to (ii) and make an arbitrage profit of \((1 + s_1)(1 + f_{1,2}) - (1 + s_2)^2\) after two years.

If \((1 + s_2)^2 > (1 + s_1)(1 + f_{1,2})\), then one can borrow $1 for one year and invest it according to (i). Then, at the end of the first year, borrow another \((1 + s_1)\) dollars to pay the first loan. This will yield an arbitrage profit of \((1 + s_2)^2 - (1 + s_1)(1 + f_{1,2})\).

Such an arbitrage scheme cannot exist in the market because many people will jump on it leading to closing the gap.

This arbitrage argument assumes that there are no transaction costs and that borrowing and lending rates are identical. This is a reasonable approximation.

In general, the forward rate \(f_{t_1, t_2}\) is the \textit{annual} interest rate charged for borrowing money between times \(t_1\) and \(t_2\), \(t_1 < t_2\).

Forward rates deduced from spot rates are termed \textit{implied forward rates} to distinguish them from \textit{market forward rates}. 
The implied forward rate between year $i$ and year $j$ satisfies 
\[(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i},\]
which implies that
\[f_{i,j} = \left[\frac{(1 + s_j)^j}{(1 + s_i)^i}\right]^{1/(j-i)} - 1.\]

For $m$ period-per-year compounding, the implied forward rate (per year) between periods $i$ and $j$ satisfies
\[(1 + s/m)^j = (1 + s_i/m)^i (1 + f_{i,j}/m)^{j-i},\]
which implies that
\[f_{i,j} = m \left[\frac{(1 + s_j/m)^j}{(1 + s_i/m)^i}\right]^{-1/(j-i)} - m.\]

Under continuous compounding, the implied forward rate (per year) between times $t_1$ and $t_2$ satisfies
\[e^{s_{t_2}} = e^{s_{t_1}} e^{f_{t_1,t_2} (t_2 - t_1)},\]
which implies that
\[f_{t_1,t_2} = \frac{s_{t_2} t_2 - s_{t_1} t_1}{t_2 - t_1}.\]

Note that (at any compounding) the spot rate at time $t$ can be seen as the forward rate between time 0 and $t$, $f_{0,t} = s_t$.

**Term structure explanations**

The spot rate curve is almost never flat but usually slopes upward.

Why is this curve not just flat at a common interest rate?

There three standard explanations for this: *Expectation theory*, *liquidity preference*, and *market segmentation*.
• **Expectation theory**
  ➢ This theory explains the shape of the spot rate curve based on expectations of what rates will be in the future.
  ➢ E.g., the theory argues that most people in the market believe that the 1-year rate next year will be higher than the current 1-year rate.
  ➢ The *expectation hypothesis* expresses this expectation in terms of forward rates.
  ➢ E.g., according to this hypothesis, the forward rate, $f_{1,2}$, is *exactly* equal to market expectation of what the 1-year rate will be next year, $s_1'$. That is, $s_1' = f_{1,2}$.
  ➢ More generally, the hypothesis is $s_{n-1}' = f_{1,n}$.
  ➢ The main weakness of expectation theory is that it implies that spot rates always increase, which is not always true.

• **Liquidity preference**
  ➢ “Liquidity” here means that most investors prefer short-term bonds to long-term ones because they are concerned about the risk of interest rate changes.
  ➢ To induce investors to buy long-term bonds, higher rates are offered on these. This leads to a rise in the spot rate curve.
• **Market segmentation**
  - The market for bonds is segmented by maturity dates.
  - That is, the group of investors buying short-term bonds is different that those buying long-term ones.
  - Then, interest rates are determined by the force of supply and demand in these distinct market segments, which leads to different interests according to maturity dates.

• **Explanation discussion**
  - Expectation theory explanation is reasonable with a deviation that may be explained by liquidity preference.

• **Expectation dynamics**
  - The expectation hypothesis leads to useful tools.
  - Spot rate forecasts: Under the expectation hypothesis, the k-year spot rate i years from now is
    \[ s_k^{(i)} = f_{i,k+i} \]
  - Specifically, if the current spot rates are \( s_0 = f_{0,1}, s_2 = f_{0,2}, \ldots, \)
    \( s_n = f_{0,n} \), then forecasts for spot rates for years 1 to n–1 are

<table>
<thead>
<tr>
<th>Year, ( i )</th>
<th>( s_1^{(i)} )</th>
<th>( s_2^{(i)} )</th>
<th>( \ldots )</th>
<th>( s_{n-2}^{(i)} )</th>
<th>( s_{n-1}^{(i)} )</th>
<th>( s_n^{(i)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( f_{0,1} )</td>
<td>( f_{0,2} )</td>
<td>( \ldots )</td>
<td>( f_{0,n-2} )</td>
<td>( f_{0,n-1} )</td>
<td>( f_{0,n} )</td>
</tr>
<tr>
<td>1</td>
<td>( f_{1,2} )</td>
<td>( f_{1,3} )</td>
<td>( \ldots )</td>
<td>( f_{1,n-1} )</td>
<td>( f_{1,n} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( f_{2,3} )</td>
<td>( f_{2,4} )</td>
<td>( \ldots )</td>
<td>( f_{2,n-2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n–2</td>
<td>( f_{n-2,n-1} )</td>
<td>( f_{n-2,n} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n–1</td>
<td>( f_{n-1,n} )</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
The discount factor between years $i$ and $j$ is

$$d_{i,j} = \left(\frac{1}{1 + f_{i,j}}\right)^{j-i}.$$ 

Note that $d_{i,k} = d_{i,j} d_{j,k}$.

- **Short rates**

  Short rates are the forward rates spanning a single time period. The short rate at year $k$ is

  $$r_k = f_{k,k+1}.$$ 

  Short rates are as fundamental as spot rates because a complete set of short rates fully defines a term structure.

  The spot rate can be obtained from short rates as follows.

  $$(1 + s_k)^k = (1 + r_0)(1 + r_1)...(1 + r_{k-1})$$

  $$\Rightarrow s_k = [(1 + r_0)(1 + r_1)...(1 + r_{k-1})]^{1/k} - 1.$$ 

  Similarly, the forward rates can be obtained from the short rates as follows.

  $$(1 + f_{i,j})^{j-i} = (1 + r_i)(1 + r_{i+1})...(1 + r_{j-1})$$

  $$\Rightarrow f_{i,j} = [(1 + r_i)(1 + r_{i+1})...(1 + r_{j-1})]^{1/j-i} - 1.$$ 

  A useful feature of short rates (under the expectation hypothesis) is that they do not change from year to year. (Spot rates do change.)

  If the short rates now are $r_0, r_1, \ldots, r_n$, then the short rates next year are $r_1, \ldots, r_n$.

  (Example 4.5 and Excel example duplicating Table 4.2)
• **Invariance theorem**
  ➢ Under the expectation hypothesis the *invariance theorem* holds.

*Theorem.* Suppose that interest evolves according to expectation dynamics. Then a sum of money invested in the interest rate market for *n* years will grow by a factor of \((1+s_n)^n\) independent of the investment strategy (as long as funds are fully invested).

➢ The invariance theorem implies that the main motivation for structuring a bond portfolio should be protection from anticipated deviation from expectation dynamics.

• **Running present value**
  ➢ This is a useful method to compute the present value of a cash flow stream under expectation dynamics.
  ➢ The present value, \(PV\), of stream \(x = (x_0, x_1, \ldots, x_n)\) is computed recursively as follows.

\[
P V(k) = x_k + d_{k,k+1}PV(k+1) = x_k +[1/(1+ r_k)]PV(k+1),
\]

\[k = n-1, n-2, \ldots, 0 .\]

where \(d_{k,k+1} = 1/(1+f_{k,k+1})\), \(PV(n) = x_n\), \(PV(0) = PV\).

➢ Running present value can be applied even if interest rates do not follow expectations dynamics as long as the short rates \(f_{k,k+1}\) could be estimated.

(Example 4.7 in Excel )
• **Floating rate bonds**
  - In a floating rate bond, the coupon payment is adjusted to match the short interest rate.
  - After every coupon payment the next coupon payment amount is *reset* to be equal to the then-current (six-month) short rate.
  - The value of a floating rate bond is equal to par at any reset point.

• **Duration under the term structure**
  - Under the term structure, duration is defined as sensitivity to linear shifts in the spot rate curve.
  - Specifically, the duration of a bond is the sensitivity of the bond value relative to $\lambda$ when the spot rates shift from $s_1, s_2, \ldots, s_n$, to $s_1 + \lambda, s_2 + \lambda, \ldots, s_n + \lambda$.
• **Fisher-Weil Duration**

➤ Under continuous compounding, the *Fisher-Weil duration* of a cash flow stream with cash flows $x_{t_i}$ at time $t_i$, $i = 1, \ldots, n$ is

$$D_{FW} = \frac{1}{PV} \sum_{i=0}^{n} t_i x_{t_i} e^{-s_i t_i},$$

where $PV = \sum_{i=0}^{n} x_{t_i} e^{-s_i t_i}$.

➤ Let $P(\lambda)$ be the value (price) of the stream when the spot rate curve shifts by $\lambda$. Then,

$$P(\lambda) = \sum_{i=0}^{n} x_{t_i} e^{-(s_i+\lambda)t_i}.$$  

(Observe that with no shift this value is $P(0) = PV$.)

➤ Upon differentiation,

$$\frac{dP(\lambda)}{d\lambda} = \sum_{i=0}^{n} -x_{t_i} t_i e^{-(s_i+\lambda)t_i}.$$  

➤ Then,

$$\left. \frac{dP(\lambda)}{d\lambda} \right|_{\lambda=0} = \sum_{i=0}^{n} -x_{t_i} t_i e^{-s_i t_i}.$$  

➤ Therefore,

$$\frac{1}{P(0)} \left[ \frac{dP(\lambda)}{d\lambda} \right]_{\lambda=0} = -D_{FW}.$$  

➤ Therefore, for a small change in the interest rate $\Delta\lambda$ the percentage decrease in bond price is $\Delta P / P = -D_{FW} \Delta \lambda$.  

• Quasi-modified duration

➢ Under discrete compounding, with \( m \) period-per-year compounding,

\[
P(\lambda) = \sum_{k=0}^{n} \frac{x_k}{[1+(s_k + \lambda)/m]^k}.
\]

➢ Then,

\[
\left[ \frac{dP(\lambda)}{d\lambda} \right]_{\lambda=0} = -\sum_{k=0}^{n} \frac{(k/m)x_k}{[1+(s_k + \lambda)/m]^{k+1}}_{\lambda=0} = -\sum_{k=0}^{n} \frac{(k/m)x_k}{[1+s_k/m]^{k+1}}.
\]

➢ Then, the quasi-modified duration is then defined as

\[
D_Q = \frac{1}{P(0)} \left[ \frac{dP(\lambda)}{d\lambda} \right]_{\lambda=0} = \frac{\sum_{k=0}^{n} (k/m)x_k (1+s_k/m)^{-(k+1)}}{\sum_{k=0}^{n} x_k (1+s_k/m)^{-k}}.
\]

• Immunization

➢ Immunization can now be done similar to Chapter 3 but by matching \( D_{FW} \) (or \( D_Q \)) instead of \( D_M \).

➢ This can be done for structuring a portfolio of bonds with different yields.

(Example 4.8)