Probability and Random Variable (1)

- Sample space and Events
  - Suppose that an experiment with an uncertain outcome is performed (e.g., rolling a die).
  - While the outcome of the experiment is not known in advance, the set of all possible outcomes is known. This set is the sample space, Ω.
  - For example, when rolling a die Ω = {1, 2, 3, 4, 5, 6}. When tossing a coin, Ω = {H, T}. When measuring lifetime of a machine (years), Ω = {1, 2, 3, ...}.
  - A subset E ⊂ Ω is known as an event.
  - E.g., when rolling a die, E = {1} is the event that one appears and F = {1, 3, 5} is the event that an odd number appears.

- Probability of an event
  - If an experiment is repeated for a number of times which is large enough, the fraction of time that event E occurs is the probability that event E occurs, P{E}.
  - E.g., when rolling a fair die, P{1} = 1/6, and P{1, 3, 5} = 3/6 = 1/2. When tossing a fair coin, P{H} = P{T} = 1/2.
  - In some cases, events are not repeated many times.
  - For such cases, probabilities can be a measure of belief (subjective probability).
• **Axioms of probability**

(1) For $E \subset \Omega$, $0 \leq P\{E\} \leq 1$;

(2) $P\{\Omega\} = 1$;

(3) For events $E_1, E_2, \ldots, E_i, \ldots$, with $E_i \subset \Omega$, $E_i \cap E_j = \varnothing$, for all $i$ and $j$, $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P\{E_i\}$.

• **Implications**

➢ The axioms of probability imply the following results:

○ For $E$ and $F \subset \Omega$,

$$P\{E \text{ “or” } F\} = P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\} ; ^1$$

○ If $E$ and $F$ are mutually exclusive (i.e., $E \cap F = \varnothing$), then

$$P\{E \cup F\} = P\{E\} + P\{F\};$$

○ For $E \subset \Omega$, let $E^c$ be the complement of $E$ (i.e., $E \cup E^c = \Omega$),

$$P\{E^c\} = 1 - P\{E\};$$

○ $P\{\varnothing\} = 0$.

• **Conditional probability**

➢ The probability that event $E$ occurs given that event $F$ has already occurred is

$$P\{E \mid F\} = \frac{P\{E \cap F\}}{P\{F\}}.$$
• **Independent events**
  - For $E$ and $F \subset \Omega$, $P\{E \cap F\} = P\{E\}P\{F\}$.
  - Two events are independent if and only if
    $$P\{E \cap F\} = P\{E\}P\{F\}.$$ That is, $P\{E|F\} = P\{E\}$.

• **Example 1**
  - Let $\Omega = \{E_1, \ldots, E_{10}\}$, where $E_i$ are mutually exclusive. It is known that $P\{E_i\} = 1/20$, $i = 1, \ldots, 6$, $P\{E_i\} = 1/5$, $i = 7, \ldots, 9$, and $P\{E_{10}\} = 3/20$.
  - Do $P\{E_i\}$ satisfy the axioms of probability?
  - No, since $\Sigma P\{E_i\} > 1$.

• **Example 2**
  - Suppose that two fair coins are tossed. What is the probability that either the first or the second coin falls heads?
  - In this example, $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. Let $E$ ($F$) be the event that the first (second) coin falls heads, $E = \{(H, H), (H, T)\}$ and $F = \{(H, H), (T, H)\}$, and $E \cap F = \{H, H\}$. The desired probability is $P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\} = 1/2 + 1/2 - 1/4 = 3/4$.

• **Example 3**
  - When rolling two fair dice, suppose the first die is 3, what is the probability the sum of the two dice is 7?
  - Let $E$ be the event that the sum of the two dice is 7, $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, and $F$ be the event that the first die is 3, $F = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$. Then,
\[ P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{P\{(3,4)\}}{P\{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}} \times 1/36}{\frac{P\{(3,1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}}{6/36}} = \frac{1}{6}. \]

- **Example 4**
  - A family has two kids. At least one of them is a boy. What is the probability that both are boys?
  - In this example, \( \Omega = \{(b, b), (b, g), (g, b), (g, g)\} \). Let \( E = \{\text{Both kids are boys}\} \) and \( F = \{\text{At least one kid is a boy}\} \). The required probability is then
  \[
  P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P\{(b, b)\}}{P\{(b, g), (g, b), (b, b)\}} = \frac{1/4}{3/4} = 1/3.
  \]

- **Example 5**
  - An urn contains three white balls and four black balls. Two balls are drawn without replacement.
  - What is the probability that the two balls are black?
  - Let \( F = \{\text{First ball is black}\} \), and \( E = \{\text{Second ball is black}\} \). The desired probability is
  \[
  P(E \cap F) = P(F)P(E \mid F) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}.
  \]
  - What is the probability that the both balls are of the same color?
  - \( P\{\text{same color}\} = P\{\text{both black}\} + P\{\text{both white}\} = \frac{2}{7} + \frac{3}{7} \times \frac{2}{6} = \frac{3}{7}. \)
• Finding Probability by Conditioning

➢ Suppose that we know the probability of event \( B \) once event \( A \) is realized (or not). We also know \( P\{A\} \). That is, we know \( P\{B|A\} \), and \( P\{B|A^c\} \) and \( P\{A\} \). What is \( P\{B\} \)?

➢ Note that

\[
B = (A \cap B) \cup (A^c \cap B) \implies P\{B\} = P\{A \cap B\} + P\{A^c \cap B\}.
\]

➢ Therefore,

\[
P\{B\} = P\{B|A\}P\{A\} + P\{B|A^c\}P\{A^c\} = P\{B|A\}P\{A\} + P\{B|A^c\}(1 - P\{A\}).
\]

➢ Here we are finding \( P\{B\} \) by “conditioning” on \( A \).

➢ In general, if the realization of \( B \) depends on a partition \( A_i \) of \( \Omega \), \( A_1 \cup A_2 \cup \ldots \cup A_n = \Omega, A_i \cap A_j = \emptyset, (i, j) \in \{1, \ldots, n\}^2, i \neq j \),

\[
P\{B\} = \sum_{i=1}^{n} P\{B|A_i\}P\{A_i\}.
\]

• Bayes’ Formula

➢ This follows from conditional probabilities. For two events,

\[
P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}} = \frac{P\{B|A\}P\{A\}}{P\{B|A\}P\{A\} + P\{B|A^c\}P\{A^c\}}.
\]

➢ With a partition \( A_i \),

\[
P\{A_j|B\} = \frac{P\{A_j \cap B\}}{P\{B\}} = \frac{P\{B|A_j\}P\{A_j\}}{\sum_{i=1}^{n} P\{B|A_i\}P\{A_i\}}.
\]
**Example 6**

- Consider two urns. The first urn contains three white and seven black balls, and the second contains five white and five black balls. We flip a coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails.

- What is the probability that a white ball is selected?

\[
P\{W\} = P\{W|H\}P\{H\} + P\{W|T\}P\{T\} = \frac{3}{10}(1/2) + \frac{5}{10}(1/2)
\]

\[
= \frac{2}{5}.
\]

- What is the probability that a black ball is selected?

\[
P\{B\} = 1 - P\{W\} = \frac{3}{5}.
\]

- What is the probability that the coin has landed heads given that a white ball is selected?

From Bayes’ formula,

\[
P\{H|W\} = \frac{P\{W|H\}P\{H\}}{P\{W\}} = \frac{\frac{3}{10}(1/2)}{\frac{2}{5}} = \frac{3}{8}.
\]

**Example 7**

- In an assembly plant, three machines, \(M_1\), \(M_2\), and \(M_3\), make 30%, 45%, and 25%, respectively of the products. It is known that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.

- What is the probability that a randomly selected product is defective?

\[
P\{D\} = P\{D|M_1\}P\{M_1\} + P\{D|M_2\}P\{M_2\} + P\{D|M_3\}P\{M_3\}
\]

\[
= (0.02)(0.3) + (0.03)(0.45) + (0.02)(0.25) = 0.0245.
\]

- If a product is found to be defective, what is the probability that it’s made by \(M_2\)?

\[
P\{M_2|D\} = \frac{P\{D|M_2\}P\{M_2\}}{P\{D\}} = \frac{(0.03)(0.45)}{0.0245} = 0.551.
\]
• **Example 8**

- In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let $p$ be the probability that she knows the answer. Assume that the student who guesses the answer will answer correctly with probability $1/m$, where $m$ is the number of multiple-choice alternatives.

- What is the probability that a student answers a question correctly?

  \[
P\{\text{Correct}\} = P\{\text{Correct|Know}\}P\{\text{Know}\} + P\{\text{Correct|Guess}\}P\{\text{Guess}\}
  \]

  \[
  = (1)(p) + (1/m)(1-p)
  \]

  \[
  = p + (1-p)/m
  \]

- E.g., if you know the answers to half of EEE questions ($m = 4, p = 1/2$), then the probability of a correct answer is $1/2 + 1/8 = 5/8$, and your likely score is $500/800$.

- What is the conditional probability that a student guessed the question given that she answered correctly?

  \[
P\{\text{Guess|Correct}\} = P\{\text{Correct|Guess}\}P\{\text{Guess}\}/P\{\text{Correct}\}
  \]

  \[
  = (1/m) (1-p)/(p + (1-p)/m)
  \]

  \[
  = (1-p)/[mp + (1-p)].
  \]

- In the EEE case, the conditional probability of guessing given a right answer is $(1/2)/(2+1/2) = 1/5$. 