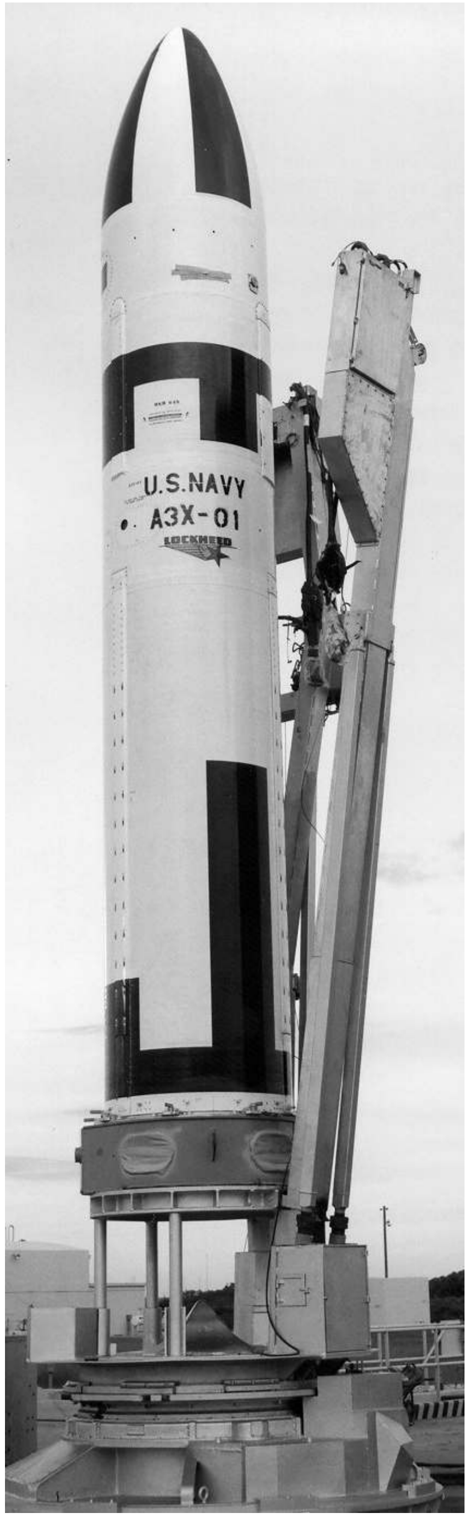
Project Management with CPM and PERT

- **Definition**
  - ➢ A project is defined by a set of *activities*.
  - ➢ Each activity is defined by its duration (time to complete the activity) and its *predecessors* (activities that must be completed before the activity can start).
  - ➢ CPM (Critical Path Method) is used to assist the project manager in scheduling the activities (i.e., when should each activity start). It assumes that activity durations are known with certainty.
  - ➢ PERT (Program Evaluation and Review Technique) is used to assist in project scheduling similar to CPM. However, PERT assumes that activity durations are random variables (i.e., probabilistic).

- **Project Network**
  - ➢ The first step in CPM/PERT is to construct a *project network*.
  - ➢ In the project network each activity is represented by an arc connected by two nodes. The first node represents the start of the activity and the second node represents the end of it.

  - ➢ The network should reflect activities precedence relations.



Unfortunately, PERT/CPM techniques were credited for the early completion of the submarine-launched **Polaris** nuclear missile.

- Given a list of activities and predecessors, the following rules should be followed to construct a project network:

(1) Node 1 represents the start of the project. An arc should lead from it to represent activities with no predecessors.

(2) A unique finish node representing the completion of the project should be included in the network.

(3) Number the nodes in such a way that the node representing completion of an activity always has a larger number than the node representing beginning of the activity.

(4) An activity should not be represented by more than one arc.

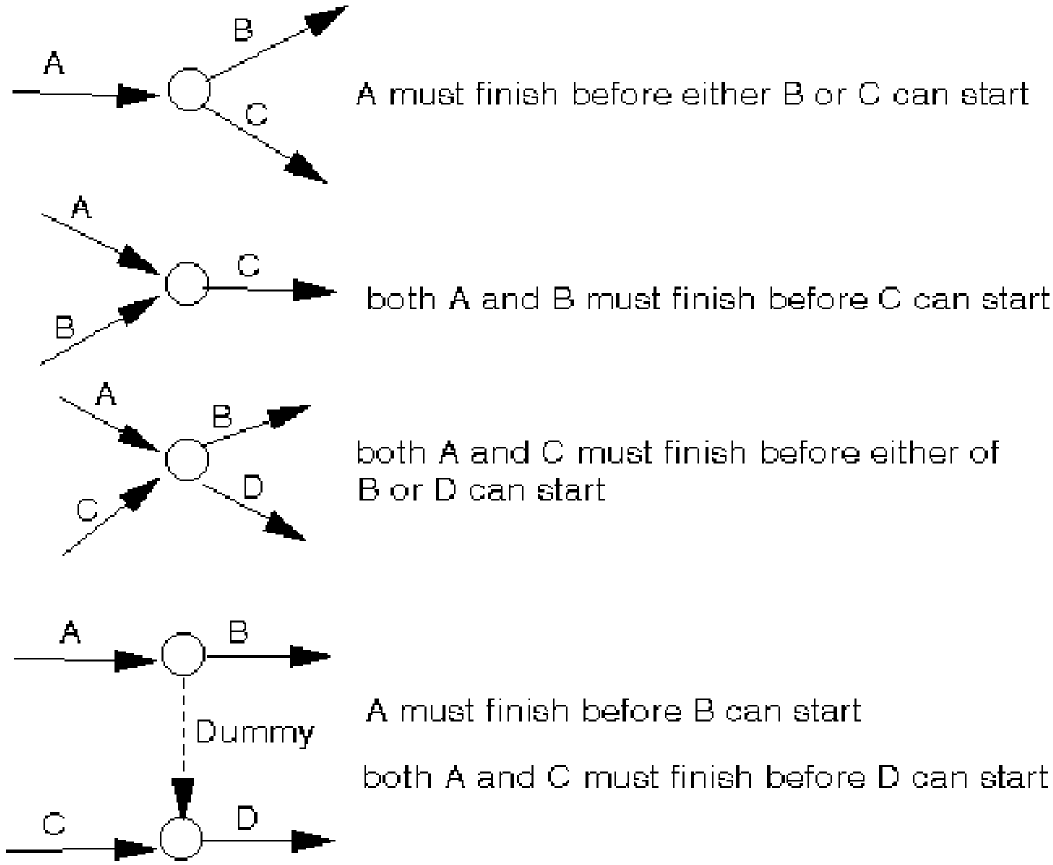
(5) Two nodes could be connected by at most one arc.

(6) Each node should have at least one entering arc and at least one leaving arc.

(7) Use the least possible number of nodes (optional).

- To avoid violation of rules (4)-(6) a dummy activity with zero duration (represented by a doted arc) may be introduced.

- Examples of network construction

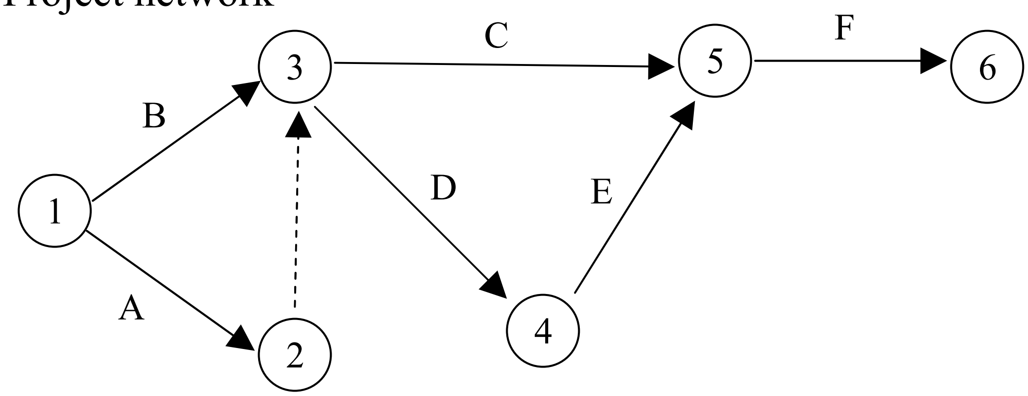


- **Example 1.**

A project to manufacture a product is composed of the following activities:

Activity	Predecessors	Duration (days)
A = train workers	--	6
B = purchase row material	--	9
C = manufacture product 1	A, B	8
D = manufacture product 2	A, B	7
E = test product 2	D	10
F = Assemble products 1 and 2	C, E	12

Project network

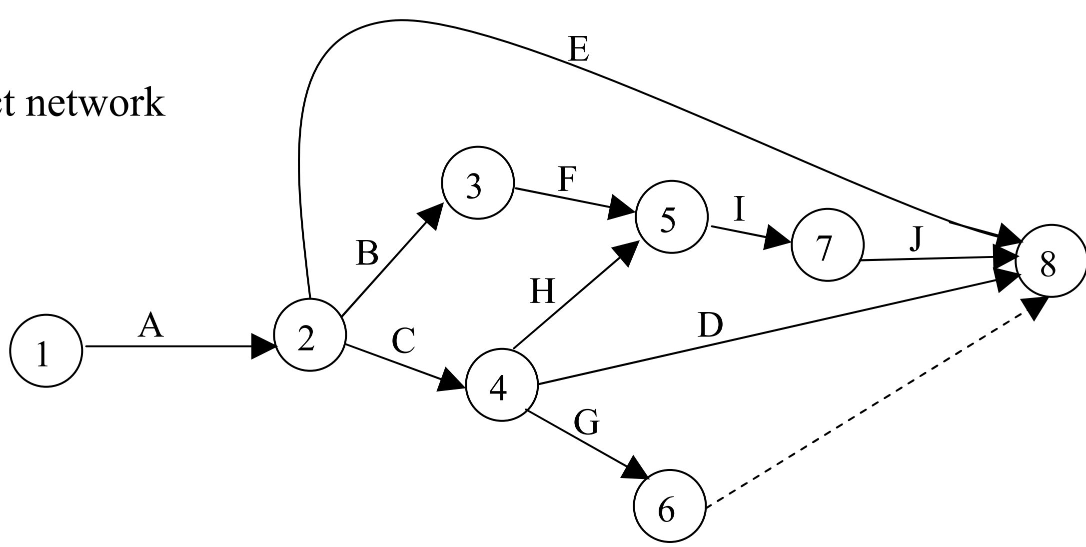


- **Example 2.**

The promoter of a rock concert must perform the following tasks before the concert can be held. Durations are in days.

Activity	Predecessors	Duration
A = find site	--	3
B = find engineers	A	2
C = hire opening act	A	6
D = set radio and TV ads	C	3
E = set up tickets agents	A	3
F = prepare electronics	B	3
G = print advertising	C	5
H = set up transportation	C	1
I = rehearsals	F, H	1.5
J = last minute details	I	2

Project network



- **Early/late event time**
  - The *early event time* for node (event) *i*, $ET(i)$ is the earliest time at which the event corresponding to node *i* can occur without violating precedence,

$$ET(i) = \max_{k \in B_i}(ET(k) + t_{ki}), \tag{1}$$

    where B_i is the set of nodes directly preceding *i* and t_{ik} is the duration of the activity with start node *k* and end node *i*.
  - Note that for the node representing beginning of the project $ET(1) = 0$. Then, $ET(i)$ is determined from (1) recursively.
  - The *late event time* for node *i*, $LT(i)$ is the latest time at which the event corresponding to *i* can occur without delaying the completion of the project,

$$LT(i) = \min_{k \in A_i}(LT(k) - t_{ik}), \tag{2}$$

    where A_i is the set of nodes directly succeeding *i*.
  - Note that for finish node, *n*, representing end of the project $LT(n) = ET(n)$. Then, $LT(i)$ is determined from (2) recursively.
  - Note also that $ET(n)$ represents the *minimum time required for the project completion* .

- **Early/late start/finish activity times**
  - Based on the early and late event times of start node i and end node j, we define the following useful quantities for activity (i, j).
  - The *early start time*, $ES(i, j)$, is the earliest time at which the activity could start, $ES(i, j) = ET(i)$.
  - The *early finish time*, $EF(i, j)$, is the earliest time at which the activity could be completed, $EF(i, j) = ET(i) + t_{ij}$.
  - The *late finish time*, $LF(i, j)$, is the latest time at which the activity could be completed without delaying the project, $LF(i, j) = LT(j)$.
  - The *late start time*, $LS(i, j)$, is the latest time at which the activity could be started without delaying the project, $LS(i, j) = LT(j) - t_{ij}$.

  Remark. Because nodes i and j can represent the start and end of several activities, $LT(i)$ and $ET(j)$, in general, have no direct physical interpretation related to activity (i, j).

- **Floats**
  - Floats are slack times by which an activity can be delayed.
  - The *total float* of an activity is the amount by which the start time of the activity can be delayed without delaying the completion of the project.

- For an activity represented by an arc connecting nodes i and j, the total float is

$$TF(i, j) = LT(j) - ET(i) - t_{ij} . \qquad (3)$$

- The *free float* of an activity is the amount by which the start time of the activity can be delayed without delaying the start of any later activity beyond its earliest starting time.
- For an activity represented by an arc connecting nodes i and j, the free float is

$$FF(i, j) = ET(j) - ET(i) - t_{ij} .$$

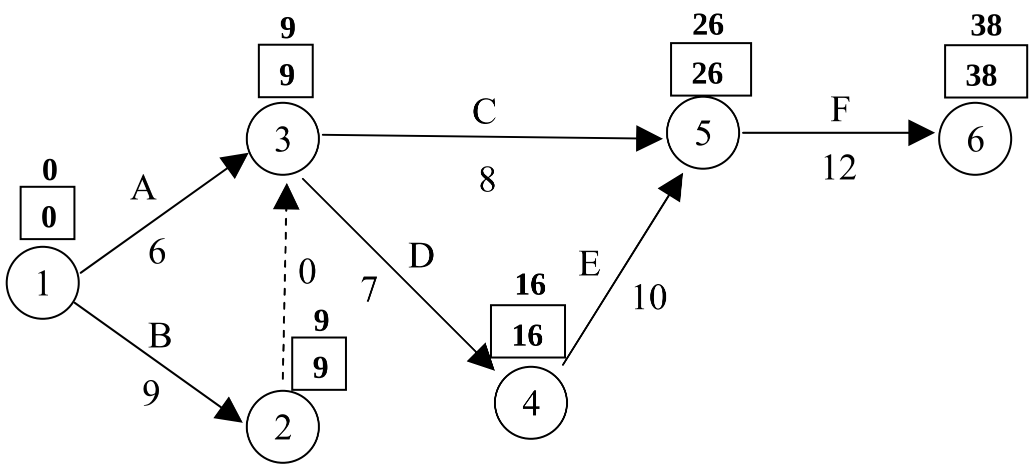
- If $FF(i, j) < TF(i, j)$, and activity (i, j) starts at time t, such that $FF(i, j) < t < TF(i,j)$, then activity (j, k) succeeding (i, j) cannot start before $ES(j) + t - FF(i, j)$.

- **Critical path**
  - A *critical activity* is an activity that cannot be delayed without delaying the completion of the project.
  - That is, a delay of Δ days on a critical activity will increase the length of the project by Δ days.
  - Critical activity should be monitored carefully to avoid delays.
  - A critical activity has a total float of zero.
  - A path from the start node to the finish node that consists entirely of critical nodes is a *critical path*.
  - A critical path is the longest path from start node to finish node.
  - The length of the critical path is the minimum time required for

project completion. It is equal to $LT(n) = ET(n)$, where n is the finish node.

- **Example 3.**

Critical path for the network in Example 1.



Starting with $ET(1) = 0$, utilizing (1) the early event times, $ET(i)$, $i = 2, \ldots, 6$, are obtained as shown in the rectangles above each node. To find the late event times, $LT(i)$, start with node 6 where $LT(6) = ET(6) = 38$. Then, utilizing (2), $LT(i)$, $i = 5, \ldots, 1$, are obtained as shown above the little squares on top of each node.

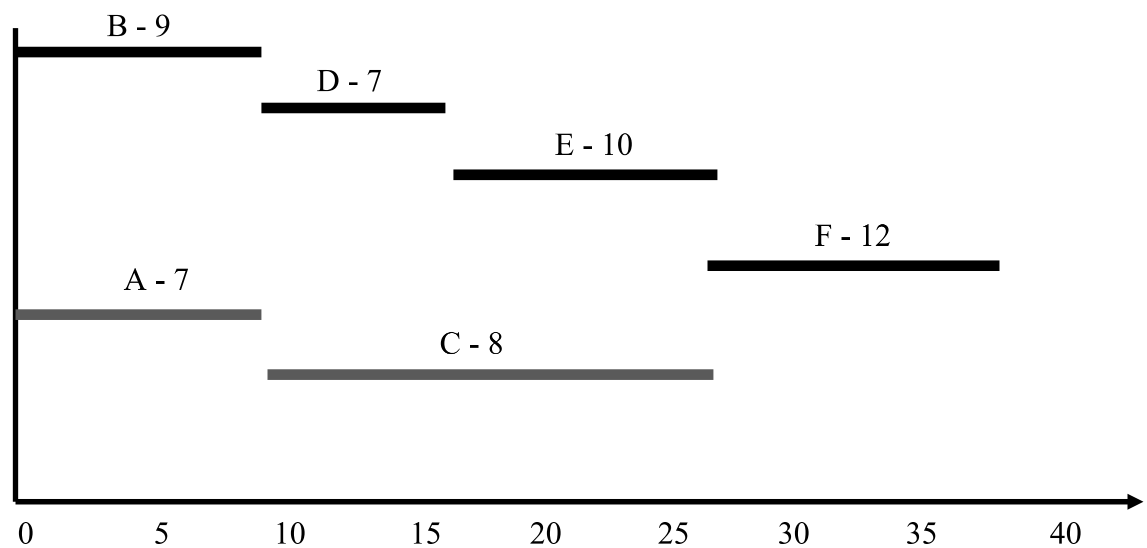
Next, the total floats for each activity, $TF(i, j)$, are evaluated from (3).

Activity (*i, j*)	**Duration (t_{ij})**	***TF*(*i, j*)**
A (1,3)	6	9 − 0 − 6 = 3
B (1, 2)	9	9 − 0 − 9 = 0
Dummy (2, 3)	0	9 − 9 − 0 = 0
C (3, 5)	8	26 – 9 – 8 = 9
D (3, 4)	7	16 – 9 – 7 = 0
E (4, 5)	10	26 – 16 – 10 = 0
F (5, 6)	12	36– 26 – 12 = 0

Critical activities are those with TF(i, j) = 0. Then, the critical path is B → D → E → F.

As examples of activity related metrics, the early start time of activity A is 0, and its latest finish time is 9. So, A could start by time 0 and must finish by time 9. Similar, C could start by time 9 and must finish by time 26.

One can construct the following schedule for this project.



- **LP formulation for finding the critical path**
  - Finding the critical path of a project network is equivalent to finding the *longest path* between the start and finish nodes.
  - Let n be the number of nodes in the network, Ω be the set of arcs (activities) in the network, A_i and B_i be as defined on page 5.
  - For $(i, j) \in \Omega$, let $x_{ij} = 1$ if (i, j) is critical, and $x_{ij} = 0$ otherwise, and t_{ij} be the duration of (i, j). Then x_{ij} are the solution of

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in\Omega} t_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j\in A_1} x_{1j} = 1 \\
& \sum_{k\in B_i} x_{ki} - \sum_{j\in A_i} x_{ij} = 0, \quad i = 2,\ldots,n-1 \\
& \sum_{k\in B_n} x_{kn} = 1 \\
& x_{ij} = 0,1, \quad (i,j)\in\Omega
\end{aligned}
$$

- The last constraint can be replaced by $x_{ij} \geq 0$, $(i, j) \in \Omega$.
- For Example 1, the LP giving the critical path is

$$
\begin{aligned}
\max \quad & Z = 6x_{12} + 9x_{13} + 0x_{23} + 7x_{34} + 8x_{35} + 10x_{45} + 12x_{56} \\
\text{subject to} \quad & x_{12} + x_{13} = 1 \\
& x_{12} - x_{23} = 0 \\
& x_{13} + x_{23} - x_{34} - x_{35} = 0 \\
& x_{34} - x_{45} = 0 \\
& x_{35} + x_{45} - x_{56} = 0 \\
& x_{56} = 1 \\
& x_{12},\, x_{13}, x_{23}, x_{34}, x_{35}, x_{45}, x_{56} \geq 0
\end{aligned}
$$

- **PERT (Program Evaluation and Review Technique)**
  - The above CPM analysis assumes that activities durations are known with certainty. This is not the case in many situations.
  - PERT analysis considers probabilistic activity duration. Specifically, duration for activity (i, j), T_{ij}, is assumed to follow a Beta distribution with parameters a_{ij}, m_{ij}, and b_{ij}.

- In practice, a_{ij}, m_{ij} and b_{ij} are estimated as the optimistic, the most likely, and the pessimistic duration of activity (i, j).
- PERT assumes that T_{ij} are independent random variable.
- Then, the length of a path P_i (i.e., the duration until event i occurs) is a random variable with mean and variance $\sum_{(k,j)\in P} E[T_{kj}]$ and $\sum_{(k,j)\in P} \text{Var}[T_{kj}]$ respectively.
- The mean and variance of T_{ij}, $E[T_{ij}]$ and $\text{Var}[T_{ij}]$, are given by

$$E[T_{ij}] = \frac{a_{ij} + 4m_{ij} + b_{ij}}{6}, \text{ and } \text{Var}[T_{ij}] = \frac{(b_{ij} - a_{ij})^2}{36}. \qquad (4)$$

- PERT assumes that the occurrence time of an event (node) in the network is the same as the length of longest path leading to the node based on *expected* durations of activities.
- PERT assumes that there are enough activities on the longest path so that the *central limit theorem* applies.
- The central limit theorem states that the sum of a "large enough" number of independent random variables approaches a Normal random variable.
- Therefore the probability that event i occurs by time t is equal probability that the length of the longest path leading to i, $L(P_i)$, is shorter than t, $P\{L(P_i) \le t\}$, where

$$P\{L(P_i) \le t\} = P\{\sum_{(k,j)\in P_i} T_{kj} \le t\} = P\{Z \le \frac{t - \sum_{(k,j)\in P_i} E[T_{kj}]}{\sqrt{\sum_{(k,j)\in P_i} \text{var}[T_{kj}]}}\}, \qquad (5)$$

where Z is the standard Normal random variable.

- In particular, the critical path is often found based on the expected activities duration.
- Then, the probability distributions of the length of the critical Path, L(CP), and of the occurrence times of events on the critical path can be found from (5).

- **Critique of PERT**
  - We have made many simplifying assumptions in PERT (Beta distribution, independence, central limit theorem).
  - Such assumptions may be unrealistic.
  - However, PERT has proven to be a quite useful tool.
  - Recent research work improves on PERT by adopting less restrictive assumptions.

- **Example 4.**

PERT analysis of the project in Example 1. Suppose now that activity durations are not known with certainty. Instead, pessimistic, most likely, and optimistic durations (in days) are as follows:

Activity (i, j)	**a_{ij}**	**m_{ij}**	**b_{ij}**	**$E[T_{ij}]$**	**Var$[T_{ij}]$**
A (1, 3)	2	6	10	6	1.778
B (1, 2)	5	9	13	9	1.778
C (3, 5)	3	8	13	8	2.778
D (3, 4)	1	7	13	7	4
E (4, 5)	8	10	12	10	0.444
F (5, 6)	9	12	15	12	1

In the following we find the probability that the project is completed within 35 days. This is equivalent to finding the probability that the length of the critical path is less or equal to 35,

$P\{L(CP) \ \le \ 35\} = P\{L(P_6) \ \le \ 35\}$.

Assuming that activity duration follows a Beta distribution, the expected values and variance of durations, $E[T_{ij}]$ and Var$[T_{ij}]$, are found from (4). Note that $E[T_{ij}]$ is the same as the deterministic durations given in Example 1. So, we can use the critical path (B → D → E → F) found in Example 3 to do PERT analysis.

Then, the expected value and variance of the critical path length are

$$E[L(CP)] = E[T_{12}] + \mathrm{E[dummy]} + E[T_{34}] + E[T_{45}] + E[T_{56}]$$
$$= 9 + 0 + 7 + 10 + 12 = 38$$

$$\mathrm{Var}[L(CP)] = \mathrm{Var}[T_{12}] + \mathrm{Var[dummy]} + \mathrm{Var}[T_{34}] + \mathrm{Var}[T_{45}] +$$
$$+ \mathrm{Var}[T_{56}] = 1.778 + 0 + 4 + 0.444 + 1 = 7.222.$$ [1]

Then,

$$P\{L(CP) \ \le \ 35\} = P\{Z \le \frac{35-38}{\sqrt{7.222}}\} \cong P\{Z \le -1.12\} = 0.131,$$

where $P\{Z \le -1.12\}$ is found using Table 1 (See also Figure 1.)

In the following we also find the probability that activity E is completed within 27 days. This is equivalent to finding the probability that the longest path leading to node 5 is less or equal to 27 days, $P\{L(P_5) \le 27\}$. Similar to the above,

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[1] Use E[dummy] = Var[dummy] = 0.

$E[L(P_5)] = E[T_{12}] + \text{E[dummy]} + E[T_{34}] + E[T_{45}]$

$= 9 + 0 + 7 + 10 = 26$

$\text{Var}[L(P_5)] = \text{Var}[T_{12}] + \text{Var[dummy]} + \text{Var}[T_{34}] + \text{Var}[T_{45}] +$

$+ \text{Var}[T_{56}] = 1.778 + 0 + 4 + 0.444 = 6.222.$

Then,

$$P\{L(P_5) \le 27\} = P\{Z \le \frac{27 - 26}{\sqrt{6.222}}\} \cong P\{Z \le 0.40\} = 0.655,$$

where $P\{Z \le 0.40\}$ is found using Table 1.

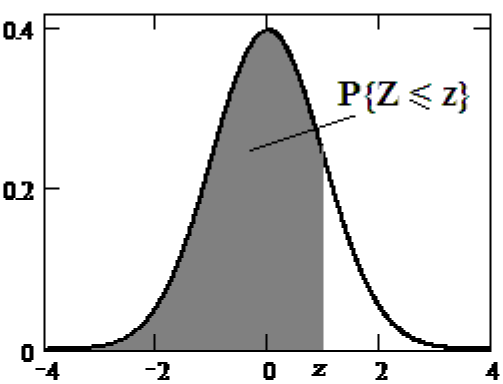

Figure 1 Area P{Z ≤ z} under the standard normal curve to the left of z

Table 1 Area $P\{Z \le z\}$ under the standard normal curve to the left of z

d	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	**0.5000**	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	**0.6554**	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	**0.8686**	0.8708	0.8729	0.8749	**0.8770**	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Examples. $P\{Z \le 0.40\} = 0.6554$,

$P\{Z < -1.12\} = 1 - P\{Z \le 1.12\} = 1 - 0.8686 = 0.1314$,

$P\{Z > 0\} = 1 - P\{Z \le 0\} = 1 - 0.5000 = 0.5000$.