Chapter 4 Nominal and Effective Interest Rates

- **Illustrative Example**
  - You placed $100 in a saving account for one year at an interest rate of 1% *per month*.
  - Calculate the amount of interest and annual interest rate.
    - \[ F = P(1 + i)^n = 100 \times 1.01^{12} = \$112.68. \]
      - The interest earned is \$112.68 - $100 = $12.68.
    - The annual interest rate is \(\frac{12.68}{100} = 12.68\%\).
  - We say that the *effective* annual interest rate is 12.68%.
  - Or, the interest rate is 12% per year, compounded monthly.
  - That is, the effective annual interest that corresponds to a 12% *nominal* annual interest, *compounded* monthly is 12.68%.

- **Nominal interest rate**
  - A nominal interest rate is an interest rate that does not include any consideration of compounding.
  - Means “in name only”, “not the true, effective rate.” E.g.,
  - 12% per year, compounded monthly
    - 12% is NOT the true effective rate (per year)
    - 12% represents the nominal rate
  - Nominal interest rate is commonly referred to as “APR” (annual percentage rate).
- **Effective interest rate**
  - Effective interest rate is the actual rate that applies for a stated period of time.
  - It takes into account the effect compounding of interest
  - Effective interest is stated in the following form:
    $$r \text{ (per year), compounded every } CP.$$ 
  - It involves two parameters
    - The annual nominal rate $r$.
    - The compounding period, $CP$, the time where interest applies
    - E.g.,
      - Daily compounding, $CP = 1 \text{ day } = 1/365 \text{ year}$.
      - Weekly compounding, $CP = 1 \text{ week } = 1/52 \text{ year}$.
      - Monthly compounding, $CP = 1 \text{ month } = 1/12 \text{ year}$.
      - Quarterly compounding, $CP = 3 \text{ months } = 1/4 \text{ year}$.
      - Semiannual compounding, $CP = 6 \text{ months } = 1/2 \text{ year}$.
  - The effective rate is called APY (annual percentage yield).

- **Factors under $m$-time-a-year compounding**
  - Under compounding over a period $CP = 1/m \text{ year}$ (e.g., $CP = 1/12 \text{ year } = 1 \text{ month}$), and at a nominal interest rate $r$, a present current $P$ is equivalent after $k$ periods (e.g. months) to
    $$F = P (1+r/m)^k \Rightarrow (P/F, r, m, k) = (1+r/m)^k .$$
  - Similarly, $F$ dollars after $k$ periods are equivalent to
    $$P = F / (1+r/m)^k \Rightarrow (F/P, r, m, k) = 1 / (1+r/m)^k .$$
• **Computing the effective interest rate**
  - Note that the effective interest rate per CP is $r/m$, where $m = 1/CP$, with CP given in fraction of a year, is the number of times interest is compounded per year.
  - E.g., with monthly compounding, $m = 12$, and a nominal rate of 12% translates into an effective monthly rate of 1% .
  - With semiannual compounding, $m = 2$, and a nominal rate of 12% translates into an effective semiannual rate of 6% .
  - Then, $1$ is equivalent to $(1 + r/m)^m$ after 1 year.
  - The effective annual rate is such that $1 + i = (1 + r/m)^m$. I.e.,
    $$i = \left(1 + \frac{r}{m}\right)^m - 1.$$

• **Continuous compounding**
  - If the compounding period, CP, is too small, $CP \to 0$, the number of compounding times gets too large, $m \to \infty$.
  - This situation is known as “continuous compounding.”
  - By noting that there are $mt$ compounding over a time $t$ expressed in years, the F/P factor over time $t$, under continuous compounding, is $\lim_{m \to \infty}(1 + r/m)^{mt} = e^{rt}$
  - Similarly, under continuous compounding, the effective annual rate is $i = e^r - 1$.
  - Continuous compounding is often assumed in quantitative finance as it simplifies the analysis.
- Example: $1 invested at a nominal rate of 8%

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<th>$m = 4$</th>
<th>$m = 12$</th>
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• **Interest rate that varies with time**
  - In practice, interest rate may vary from one period to the other.
  - In particular, it is often *expected* that the interest rate will *increase* with time.
  - If the interest rates in periods, 1, ..., n are \( i_1, \ldots, i_n \). Then, the future worth after n periods, \( F \), of a present amount \( P \) is
    \[
    F = P(1+i_1)(1+i_2)\ldots(1+i_n) = P \prod_{i=1}^{n} (1+i_i).
    \]
  - The rates \( i_1, \ldots, i_n \) are known as *short rates*.
  - The short rate \( i_t \) represents the expected 1-year rate after \( t \) years.