Chapter 4 Nominal and Effective Interest Rates

• Example
  ➢ You placed $100 in a saving account for one year at an interest rate of 1% per month.
  ➢ Calculate the amount of interest and annual interest rate.
    ○ $F = P(1+i)^n = 100 \times 1.01^{12} = $112.68$. The interest earned is $112.68 - 100 = $12.68$.
    ○ The annual interest rate is $12.68/100 = 12.68\%$.
  ➢ We say that the effective annual interest rate is $12.68\%$.
  ➢ Or, the interest rate is $12\%$ per year, compounded monthly.
  ➢ That is, the effective annual interest that corresponds to a $12\%$ nominal annual interest, compounded monthly is $12.68\%$.

• Nominal interest rate
  ➢ A nominal interest rate is an interest rate that does not include any consideration of compounding.
  ➢ Means “in name only”, “not the true, effective rate.” E.g.,
  ➢ 12% per year, compounded monthly
    ○ 12% is NOT the true effective rate (per year)
    ○ 12% represents the nominal rate
  ➢ Nominal interest rate is commonly referred to as “APR” (annual percentage rate).
• **Effective interest rate**
  - Effective interest rate is the actual rate that applies for a stated period of time.
  - It takes into account the effect compounding of interest.
  - Effective interest is stated in the following form:
    
    \[ r \text{ (per year), compounded every } CP. \]
  - It involves two parameters
    - The annual nominal rate \( r \).
    - The compounding period, \( CP \), the time where interest applies
    - E.g.,
      - Daily compounding, \( CP = 1 \text{ day } = 1/365 \text{ year} \).
      - Weekly compounding, \( CP = 1 \text{ week } = 1/52 \text{ year} \).
      - Monthly compounding, \( CP = 1 \text{ month } = 1/12 \text{ year} \).
      - Quarterly compounding, \( CP = 3 \text{ months } = 1/4 \text{ year} \).
      - Semiannual compounding, \( CP = 6 \text{ months } = 1/2 \text{ year} \).
  - The effective rate is called APY (annual percentage yield).

• **Factors under \( m \)-time-a-year compounding**
  - Under compounding over a period \( CP = 1/m \) year (e.g., \( CP = 1/12 \text{ year } = 1 \text{ month} \)), and at a nominal interest rate \( r \), a present current \( P \) is equivalent after \( k \) periods (e.g. months) to
    
    \[ F = P(1+r/m)^k \Rightarrow (P/F, r, m, k) = (1+r/m)^k. \]
  - Similarly, \( F \) dollars after \( k \) periods are equivalent to
    
    \[ P = F / (1+r/m)^k \Rightarrow (F/P, r, m, k) = 1 / (1+r/m)^k. \]
• Computing the effective interest rate
  
  ➢ Note that the effective interest rate per CP is $r/m$, where $m = 1/CP$, with $CP$ given in fraction of a year, is the number of times interest is compounded per year.
  
  ➢ E.g., with monthly compounding, $m = 12$, and a nominal rate of 12% translates into an effective monthly rate of 1% .
  
  ➢ With semiannual compounding, $m = 2$, and a nominal rate of 12% translates into an effective semiannual rate of 1% .
  
  ➢ Then, $1$ is equivalent to $(1 + r/m)^m$ after 1 year.
  
  ➢ The effective annual rate is such that $1 + i = (1 + r/m)^m$. I.e.,

$$i = \left(1 + \frac{r}{m}\right)^m - 1.$$  

• Continuous compounding
  
  ➢ If the compounding period, $CP$, is too small, $CP \to 0$, the number of compounding times gets too large, $m \to \infty$.
  
  ➢ This situation is known as “continuous compounding.”
  
  ➢ Under continuous compounding, the rate of growth of an investment over a time $t$ expressed in years, i.e., the P/F factor, is

$$\lim_{m \to \infty} (1 + r/m)^mt = e^r.$$  

  ➢ Similarly, under continuous compounding, the effective rate is $i = e^r - 1$. 
Continuous compounding is often assumed in quantitative finance as it simplifies the analysis.

- **Example: $1 invested at a nominal rate of 8%**

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<th>Year</th>
<th>$m = 1$</th>
<th>$m = 2$</th>
<th>$m = 4$</th>
<th>$m = 12$</th>
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• Bonds
  ➢ Bonds represent the major source that governments and companies use to obtain debt financing.
  ➢ A bond is an obligation by the bond issuer to pay money to the bond holder (buyer).
  ➢ A bond pays its face value or par value at its maturity date.
    In addition, bonds usually pay periodic coupon payments. In the U.S., coupon payments are made every 6 months.
  ➢ The coupon amount is described in percent of face value.
  ➢ For example, a 10% coupon with a face value of $1000 will pay $100 coupon per year. If payment is semiannual, the coupon payment will be $50.
  ➢ Usually coupon rates are close to the prevailing interest rate.
  ➢ A bond can be traded freely in the market place. Its price varies continuously.
  ➢ A bond’s yield to maturity is the interest rate at which the PV of coupon and face value payments are equal to the bond price. This is always quoted on an annual basis.
  ➢ Yield to maturity (YTM) is actually the internal rate of return (IRR) of the bond (see Chapter 7).
  ➢ Consider a bond with a price of $P$ and a face value $F$, making $m$ coupon payments per year of $C/m$ (with a total of $n$ payments).
the YTM is the value of \( \lambda \) such that

\[
P = \frac{F}{(1 + \lambda/m)^n} + \sum_{k=1}^{n} \frac{C/m}{[1 + (\lambda/m)]^k}.
\]

This formula assumes that the interest is compounded every payment period.

Upon simplification,

\[
P = \frac{F}{[1 + (\lambda/m)]^n} + \frac{C}{\lambda} \left(1 - \frac{1}{[1 + (\lambda/m)]^n}\right).
\]

This formula implies that the price of the bond is decreasing in its yield. That is, a high-yield bond will have a “low” price (this is the investor’s point of view).

Bond yields are quoted in the financial media.

E.g., Lebanese treasury bills yield (source: BLOM brief)

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### Treasury Yields

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<th>Change bps</th>
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<td>6-M TB yield</td>
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<td>7.17%</td>
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<td>12-M TB yield</td>
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<td>7.66%</td>
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<td>24-M TB coupon</td>
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<td>8.38%</td>
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<tr>
<td>60-M TB coupon</td>
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### Treasury Yields

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<td>6-M TB yield</td>
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<tr>
<td>12-M TB yield</td>
<td>5.08%</td>
<td>5.08%</td>
<td>0</td>
</tr>
<tr>
<td>24-M TB coupon</td>
<td>5.84%</td>
<td>5.84%</td>
<td>0</td>
</tr>
<tr>
<td>36-M TB coupon</td>
<td>6.50%</td>
<td>6.50%</td>
<td>0</td>
</tr>
<tr>
<td>60-M TB coupon</td>
<td>6.74%</td>
<td>6.74%</td>
<td>0</td>
</tr>
</tbody>
</table>

### Bond example

- What is the price of a 10% (coupon, paid semiannually), 30-year US Treasury bond with yield 4%? Assume a face value of 100. (Bond prices are typically quoted as percentage of face value.)

- In this example, \( F = \$100 \), \( C = 0.1 \times 100 = \$10 \), and \( \lambda = 4\% \), \( n = 30 \times 2 = 60 \).

\[
P = \frac{F}{[1 + (\lambda / m)]^n} + \frac{C}{\lambda \left(1 - \frac{1}{[1 + (\lambda / m)]^n}\right)}
\]

\[
= \frac{100}{(1 + 0.04/2)^{60}} + \frac{10}{0.04 \left(1 - \frac{1}{(1 + 0.04/2)^{60}}\right)} = \$204.28
\]

### Interest rate that varies with time

- In practice, interest rate may vary from one period to the other.

- In particular, it is often expected that the interest rate will increase with time.
This fact is reflected in the annual yield on long (maturity) bonds being higher than that on short bonds. (See Lebanese TB above.)

If the interest rates in periods, 1, ..., n are $i_1, ..., i_n$. Then, the future worth after n periods, $F$, of a present amount $P$ is

$$F = P(1+i_1)(1+i_2)...(1+i_n) = P \prod_{t=1}^{n} (1+i_t).$$

The rates $i_1, ..., i_n$ are known as short rates.

The short rate $i_t$ represents the expected 1-year rate after $t$ years.

Short rates are estimated based on the yields of bonds similar to those above.