Chapter 2 Factors: How time and interest affect money

• Single Payment Factors
  ➢ Recall that $P$ dollars now are equivalent to $F$ dollars after $n$ time periods at an interest rate of $i$ per time period$^1$, where
  \[ F = P(1+i)^n. \]
  ➢ Rewrite this as
  \[ F = P \times (F/P, i, n), \]
  where $(F/P, i, n) = (1+i)^n$ is the “$F/P$ factor.”
  ➢ In addition, this implies that
  \[ P = \frac{F}{(1+i)^n} = F \times (P/F, i, n), \]
  where $(P/F, i, n) = \frac{1}{(1+i)^n}$ is the “$P/F$ factor.”

• Tables and Spreadsheets
  ➢ The $P/F$ and $F/P$ factors, as well as other factors, are tabulated at the end of your textbook.
  ➢ You may use these tables or the formulas that we will derive.
  ➢ Spreadsheets (i.e., Excel) has built-in function for factors (or you can easily build your own functions) for calculating the factors. Excel is very suited for practical interest calculations.

$^1$ Here and elsewhere, when the type of interest is not specified, assume it’s compound interest.
• **Uniform Series Factors**

➤ Suppose one will pay $A$ dollars every time period for $n$ period starting with the end of period 1 (see figure below).

![Diagram of uniform series payments](image)

➤ Then, this series of cash flows is now equivalent to

\[
P = \frac{A}{1+i} + \frac{A}{(1+i)^2} + \ldots + \frac{A}{(1+i)^n} = A \sum_{j=0}^{n-1} \left( \frac{1}{1+i} \right)^j = A \frac{1 - \left( \frac{1}{1+i} \right)^n}{1 - \frac{1}{1+i}}.
\]

➤ Upon simplification,

\[
P = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right] = A \frac{1 - \frac{1}{(1+i)^n}}{1 - \frac{1}{1+i}} = A \times (P/A, i, n).
\]

➤ \((P/A, i, n) = [(1+i)^n - 1]/[i(1+i)^n]\) is the “P/A factor.”

➤ In addition,

\[
A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] = P \times (A/P, i, n),
\]

➤ \((A/P, i, n) = i(1+i)^n / [(1+i)^n - 1]\) is the “A/P factor.”

➤ Finally, to find the future amount, $F$, equivalent to the uniform series of cash
\[ F = P(1+i)^n = A \left( \frac{(1+i)^n - 1}{i} \right) = A \times \left( \frac{F}{A}, i, n \right), \]
\[ A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] = \frac{F}{(1+i)^n} \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] = \frac{i}{(1+i)^n - 1} \]
\[ = F \times \left( \frac{A}{F}, i, n \right), \]

- \((F/A, i, n) = \left( (1+i)^n - 1 \right)/i\) is the “F/A factor.”
- \((A/F, i, n) = i/((1+i)^n - 1)\) is the “A/F factor.”

**Running amortization**

- Amortization is the process of substituting a current payment \( P \) for periodic payments of \( A \) per period (e.g. car or home loan.)
- One can view each amortization payment \( A \) as composed of two parts: (i) interest on running (outstanding) balance and (ii) partial repayment of principal.
- This procedure is equivalent to re-amortizing the running balance every period over the remaining time horizon.
- This is consistent with accounting practice.
- E.g., consider a loan of $1,000 issued on Jan 1, 2006, to be paid back in equal monthly payments over 5 years at an interest rate of 1% per month.
- The monthly payment is
\[ A = 1000 \left[ \frac{(0.01)(1+0.01)^60}{(1+0.01)^60 - 1} \right] = $22.24 . \]
Then, the outstanding balance on Feb 1, 2006 is $1,000 minus the monthly payment ($22.24) plus the monthly interest (0.01×1,000=$10), which gives $987.76.

The (running) amortization of the $987.76 at 1-Mar-2006 over the remaining 59 months is

\[ A = 987.67 \left[ \frac{(0.01)(1+0.01)^{59}}{(1+0.01)^{59}-1} \right] = 22.24 \text{.} \]

The (running) amortization of the $975.39 at 1-Apr-2006 over the remaining 58 months is also $22.24, and so on.

<table>
<thead>
<tr>
<th>Date</th>
<th>Previous balance</th>
<th>Interest</th>
<th>Payment Received</th>
<th>New Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-Jan-06</td>
<td>$1,000</td>
<td></td>
<td></td>
<td>$1,000</td>
</tr>
<tr>
<td>1-Feb-06</td>
<td>$1,000</td>
<td>$10.00</td>
<td>$22.24</td>
<td>$987.76</td>
</tr>
<tr>
<td>1-Mar-06</td>
<td>$987.76</td>
<td>$9.88</td>
<td>$22.24</td>
<td>$975.39</td>
</tr>
<tr>
<td>1-Apr-06</td>
<td>$975.39</td>
<td>$9.75</td>
<td>$22.24</td>
<td>$962.90</td>
</tr>
<tr>
<td>1-May-06</td>
<td>$962.90</td>
<td>$9.63</td>
<td>$22.24</td>
<td>$950.28</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1-Dec-10</td>
<td>$43.83</td>
<td>$0.44</td>
<td>$22.24</td>
<td>$22.02</td>
</tr>
<tr>
<td>1-Jan-11</td>
<td>$22.02</td>
<td>$0.22</td>
<td>$22.24</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
Arithmetic Gradient Factors

- In some cases cash flows increase by a fixed amount in each time period starting with period 2.
- Starting with a cash flow of $A$ at the end of period 1, the cash flows increase by the gradient, $G$, in each period. That is, the cash flows are $A, A + G, A + 2G, \ldots, A + nG$, in periods 1, 2, ..., $n$.
- Then, at time 0, this series of cash flows is equivalent to
  \[
P = P_A + P_G,
  \]
  where $P_A$ is the equivalent at time zero of the series with uniform cash flows $A$ per time period,
  \[
P_A = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right].
  \]
- $P_G$ is the equivalent at time zero of the arithmetic cash flow series with gradient $G$ (i.e., the series having $G, 2G, \ldots, (n-1)G$ cash flows at the end of periods 2, 3, ..., $n$.)
\( P_G \) is evaluated as follows

\[
P_G = \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \ldots + \frac{(n-2)G}{(1+i)^{n-1}} + \frac{(n-1)G}{(1+i)^n}
\]

\[
= \frac{G}{1+i} \sum_{j=1}^{n-1} \frac{j}{(1+i)^j} = \frac{G}{1+i} \left( \frac{(1+i) - [1 + i + (n-1)i]/(1+i)^{n-1}}{i^2} \right)
\]

\[
= \frac{G}{i} \left( \frac{(1+i)^n - (1+ni)}{i(1+i)^n} \right) = \frac{G}{i} \left( \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right) = G(\frac{P}{G}, i, n).
\]

\( (\frac{P}{G}, i, n) = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \) is the "P/G factor".

In the above we have used the fact, that for \( j \) and \( m \) integers

\[
\sum_{j=1}^{m} \frac{j}{(1+y)^j} = \frac{(1+y) - (1+y+my)/(1+y)^m}{y^2}.
\]

Finally, we define “A/G” and “F/G” factors.

**Geometric Gradient Factors**

Suppose now that in a series of a cash flows the amounts increase (or decrease) by a fixed amount \((1+g)\), in each time period starting with period 2.
At time 0, this geometric gradient series is equivalent to

\[ P_g = \frac{A_1}{1+i} + \frac{A_1(1+g)}{(1+i)^2} + \ldots + \frac{A_1(1+g)^{n-1}}{(1+i)^n} = \frac{A_1}{(1+i)} \sum_{j=0}^{n-1} \left( \frac{1+g}{1+i} \right)^j. \]

It follows that

\[ P_g = \begin{cases} 
A_1 \left\{ \frac{1 - [(1+g)/(1+i)]^n}{i-g} \right\}, & \text{if } i \neq g, \\
\frac{nA_1}{1+i}, & \text{if } i = g.
\end{cases} \]

**Summary of terminology**

- **F/P** factor: Compound Amount Factor
- **P/F** factor: Present Worth Factor
- **P/A** factor: Uniform-Series Present Worth Factor
- **A/P** factor: Capital Recovery Factor
- **A/F** factor: Sinking Fund Factor
- **F/A** factor: Uniform-Series Compound Amount Factor
- **P/G** factor: Arithmetic gradient present worth factor
- **A/G** factor: Arithmetic gradient uniform-series factor